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Department of Mathematical Sciences

Ph. D. Thesis

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Doctor of Philosophy

Theoretical and Numerical Study on Optimal Mortgage Refinancing Strategy

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Abstract

This work studies optimal refinancing strategy for the debtors on the view of balancing the profit and risk, where the strategy could be formulated as the utility optimization problem consisting of the expectation and variance of the discounted profit if refinancing. An explicit solution is given if the dynamic of the interest rate follows the affine model with zero-coupon bond price. The results provide some references to the debtors in dealing with refinancing by predicting the value of the contract in the future. Special cases are considered when the interest rates are deterministic functions. Our formulation is robust and applicable to all of the short rate stochastic processes satisfying the affine models.

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Chapter 1

Introduction

As one of the most frequently traded financial instruments, mortgage contract provides its debtors a way to manage their accounts. Valuation of mortgage security is of pivotal importance to investors, bankers and brokers in helping with their decision making from various perspectives. Knowledge of this kind is used as a key economic indicator not only in developed markets such as the US market, but also increasingly in emerging markets such as China and Brazil (see Lynn et. al [25]). The valuation of mortgage securities has to take into account the contracted choices to the debtors, among which refinancing is one of the most common choices. The main financial reason leading to refinancing, not taking into consideration the socioeconomic factors, is to take advantage of lower interest rate. There has been a great deal of research on the topic of modelling mortgage refinancing behaviours (see for example, Chen and Ling [9], Dunn and McConnell [13, 14], Lee and Rosenfield [21], Longstaff [23]).

1.1 Research Objectives and Contributions

This work studies the optimal strategy for the debtors on the refinancing questions based on continuous payment. In our study, refinancing is described as a financial behavior to replace the existing mortgage contract with a new contract. The initial balance of the new contract is the outstanding balance of the original contract, and the duration of the new contract is the remaining duration of the original contract. This research has two objectives. The first is to propose an analytical model to capture the value of the contract, which can also be described as the profit that the debtor and the lender (i.e. financial institutions) may gain from refinancing, and point out the time where the expected value of the contract can be maximized. The second objective is to find the balance or the tradeoff between the profit and the risk of refinancing with a utility function. The optimal time under such a utility will be influenced both by the profit and the risk. We remark that the implementation of our approach does not restrict to the choice of any affine stochastic model, as long as such a model explains the market trend with acceptable significance.

This thesis presents a number of original contributions to the field of Mathematical Finance. These include:

1. An extension of the previous assumptions of constant discount factor (see Lee and Rosenfield [21], Dunn and McConnel[13], [14] et. al) by suggesting that the discount factor is a stochastic process.
2. Presenting general models for the expected value and variance of the portfolio consisted of a loan and a refinancing agreement, which is applicable to all the affine interest rate models.
3. Redefining the optimal time to mortgage refinancing including risk.
4. A new asymptotic analysis for the expected value and variance of the portfolio.
5. Proposing a utility function approach describing the tradeoff between profit and risk, which makes the problem more realistic and applicable.

1.2 Organization of this Thesis

The contents of each chapter are outlined as follows.

Chapter 2: Preliminaries

We describe various views to capture the payment behavior of the debtors, and the methods that have been adopted to solve, either analytically or numerically, the value of the mortgage contract, or to make the refinancing decisions. First, we describe the approach named structure-form method, and then we represent the reduced-form approach. In addition, the term-structure of interest rates models are reviewed in this chapter.

Chapter 3: Modelling of Refinancing

We formulate some assumptions to support our methodology. To compromise the risk and the profit of refinancing, a utility function approach is proposed to describe the satisfaction of the debtors. In addition, we present general formulations to capture the expectation and the variance of the value of the contract.

Chapter 4: Results for Various Models

Our formulations are applied to three of the most common affine short rate models, Merton model, Vasicek model and CIR model. Various methods are adopted to obtain the asymptotic solution of the risk and the expected profit. In addition, we make comments on the numerical results based on these three models.

Chapter 5: Special case: $\sigma = 0$

We consider the special case that the volatility of the market interest rate is zero. We focus on the value of the contract and the optimal time the debtor may want to refinance. Variation analysis of the life of the contract is presented in this chapter.

Chapter 6: Remarks and Future Work

We present some future work and directions for this research and summarize our project and point out the merits of our method for research in Mathematical Finance.

Chapter 2

Preliminaries

2.1 General Introduction

The pricing of mortgages in the context of stochastic interest rate plays an important role for financial management. The contributing factors impacting the value of mortgage contract have been explored by abundant literatures. As one of the most influential financial instruments in both the primary and secondary market, residential mortgage contract typically grants the debtor several options to facilitate his or he reaction to the market movement, among which the option of refinancing is of pivotal importance. In fact, a rather more common scenario in China's market is that the majority of mortgage debtors make periodical mortgage payment using their fixed income inflow from other sources, typically in the form of salary, for instance. This economy reality underscores the importance of the option of refinancing. (see Zheng et. al[48])

There has been a great deal of research on the topic of modelling mortgage refinance and prepayment behaviours. These works endeavoured to understand the conditions under which a debtor will pay back his or her outstanding debt before the end of the contracted period. Our motivation is different from most of the earlier work modelling the optimal mortgage prepayment problem. Their purpose was to determine the fair price of a mortgage contract under the condition that the loan may be prepaid or default. This mortgage contract pricing problem is closely related to the valuation of residential mortgage backed securities (MBSs) – an important problem as the MBS market has been one of the largest and fast-growing bond markets in the United States. One approach to the mortgage contract pricing is to view the prepayment or default opportunity as a built-in option in the mortgage contract that can be exercised by the debtor under favourable conditions. This approach inevitably borrows techniques from option pricing to calculate prices of mortgage contracts.

In an important early work, Dunn and McConnel ([13], [14]) first applied the contingent claim techniques to estimate the present value of the mortgage backed pass-through securities, in which partial differential equations were constructed and solved using the finite-difference method. Following the option pricing approach, Chen and Ling ([9])

applied the binomial tree method to calculate the prices of the prepayment option and the mortgage contract. They considered the fixed-rate mortgage contract and assumed that a debtor would prepay the outstanding debt when the contract rate dropped deep enough. Their model incorporated the possibility of recursive refinancing. However, the optimal refinancing threshold rate (the rate under which refinancing, if takes place, will be optimal) cannot be obtained directly from the binomial tree. The optimal refinancing threshold rates could only be approximated through multiple tests. The difference in basis point between this rate and the original contract rate was deemed as the value that the mortgage rate has to drop to make refinancing at present time optimal. The Longstaff-Schwartz ([24]) least-square Monte Carlo method is a well-known approach in option pricing to value the prices of multi-asset American options. In [23], Longstaff used this method to compute the prices of the prepayment options and the mortgage contracts. More recently, Lee and Rosenfield ([21]) applied dynamic programming technique to estimate the overall cost to the debtor refinancing the outstanding debt at a particular time with a new mortgage rate. The authors assumed that refinancing would happen if this cost was lower than the overall cost without refinancing.

A wide variety of approaches have been applied to the refinancing problem, most of which can be categorized into two main areas, as summarized by Pliska ([30]). One category is called option-based or structural approach, in which the termination behaviour is modelled as the optimal response of a rational debtor to the changes of some potential state variables, such as mortgage interest rate and house price. This type of model is closely related to value the early exercise feature of American options. The previous literature applies the contingent claim techniques to minimise the present value of the mortgage contract. A rational debtor will compare the liability and outstanding principal to make decisions of immediate refinancing or postponing for an additional period. Some researchers who followed and extended the option-based method are Schwartz and Torous ([8]), Dunn and Spatt ([15]), Timmis ([38]) , Johnston and Drunen ([17]), Kau et al ([19], [20]). The second main category is called a reduced form approach, an exogenous approach, an empirical approach and an econometric approach. The reduced form approach usually builds a statistical model demonstrating how the value of mortgages relies on interest rates and possibly other related factors. This method assumes that prepayment time is a random time governed by some hazard rate to be estimated from the historical prepayment data in large mortgage pools. Schwartz and Torous ([31]) first introduced the concept of hazard rate describing the random time for prepayment and formulated a partial differential equation for the value of a mortgage contract through a two-factor model. Recent developments involving the hazard rate as a function of a default time were presented in the papers of Schwartz and Torous([32], [33]), Deng ([11]) and Deng et al ([12]).

2.2 Basic Concept of Mortgage Contract

2.2.1 The behaviour of the debtors

A mortgage is a type of legal agreement that conveys the conditional right of ownership on an asset or property by its owner (the debtor) to a lender (the mortgagee) as security for a loan. Virtually any legally owned property can be mortgaged, although real properties (land and buildings) are the most common (see [2]). Mortgage contracts typically carry a lower interest rate than other loans since the real property can act as a collateral. If the debtor suffers a worse financial condition and cannot afford to repay the loan, the lenders have the right to take over the assets, which is called default. The debtor also has the right to terminate the contact, the behavior of which is called prepayment or refinancing. This thesis will concentrate on the typical case of refinancing.

Prepayment refers to that behavior that the debtor chooses to settle all or part of the loan balances even though the lender's preference may be to keep receiving the contracted continuous or periodical instalments, depending how the loan interest is collected (of course, real continuous collection of interest is not possible in banking practice)(see [43]). The main financial reason leading to prepayment, not taking into consideration socioeconomic factors, is typically the low investment return that the debtor may earn using the money at hand. That is, the available investment return for the debtor, on average, does not compensate his contracted continuous payment pledges to the lender. The studies on this aspect have seen important development recently, especially those contained in the paper of Xie et al ([46]), and Xie ([44], [45]), for instance, where the combination of advanced mathematical analysis with novelty numerical methods has made it possible to find very fast and cost effective solutions to the problem when the underlying interest rate is assumed as a specific but commonly adapted mean reverting model. (see Zheng et. al[48])

On the other hand, not all debtors have sufficient fund to make alternative investment. The main reason for debtors to refinance is to improve the financial leverage efficiency by obtaining an alternative mortgage loan with a lower interest rate. Most of the previous literatures in this topic are empirical in nature from the perspective of optimal refinancing differentials, where the optimal differential is defined when the net present value of the interest payment saved reaches the sum of refinancing costs (see the paper of Agarwal et al ([3]) and relevant references contained therein). (see Zheng et. al[48])

The behaviour of terminating the original contract can occur for financial reasons or other exogenous variables. The exogenous reasons will affect the value of the mortgage payment indirectly, such as divorce or moving. For example, if the debtor knows that he or she is likely to move, hence he or she might terminate the contract on some day in the future. Relative to the case, it is more attractive to terminate the contract immediately than to wait for an optimal time. In our research, early termination is only considered for endogenous or financial reasons, based on the assumption that the value of the mortgage to the bank is equal to the total debt the debtor has to repay.

2.2.2 Two basic types of mortgage contract

Some of the following paragraphs are from ([1], [42]).

The two basic types of amortized loans are the fixed rate mortgage (FRM) and adjustable-rate mortgage (ARM). Fixed rate mortgages are prevalent because they allow the debtor to predict what the payments will be in the future over the duration of the loan. No matter what happens with interest rates, the payments won't change if he or she has been involved in a fixed rate mortgage. This contrasts to the adjustable rate mortgages who do not have a fixed rate, leaving the debtor vulnerable and dependent upon the interest rate, which changes periodically. With a fixed rate mortgage, the debtor can calculate the amount of monthly payment, and the time he or she can pay off all the principal and interest. He or she will pay the same monthly payment during the life of the fixed rate mortgage contract. The monthly payment consists of three components, the fraction of principal balance, the interest rate payment and the transaction cost, or the service fee if the debtor wants to terminate the contract. The monthly payment in the fixed rate contract is higher than other mortgage choices, such as the fixed rate mortgage which offers the safety of knowing that the future payments will not increase.

The fixed rate mortgage is practical as it will not affect the debtor, if the rates increase. If the interest rates happen to decrease, it still will not affect the debtor as he or she can decide to refinance the loan to benefit from a better interest rate. An adjustable-rate mortgage differs from a fixed-rate mortgage in many ways. Most importantly, with a fixed-rate mortgage, the interest rate stays the same during the life of the loan. With an ARM, the interest rate changes periodically, usually in relation to an index, and payments may go up or down accordingly. The rate for an adjustable rate mortgage is determined by some market indices. Many adjustable rate mortgages are tied to the LIBOR, Prime rate, Cost of Funds Index, or other indices. A main reason to consider adjustable rate mortgages is that the debtor may end up with a lower monthly payment. The bank rewards him or her with a lower initial rate because the debtor is taking the risk that interest rates could rise in the future. However, the increase in mortgage payments can be significant if interest rates rise. Some debtors are unprepared for the increase in mortgage payments, and they may find themselves in dire financial straits when mortgage payments increase unexpectedly.

The thesis will only concentrate on fixed-rate mortgage contract, which is the most popular one in United States that almost 75% of all home loans are fixed rate mortgage.

In financial terms, most literature considered the behaviour of the prepayment right which can be viewed as an American option. The debtor can prepay the loan at any time during the period of the contract. Compared to prepayment, refinancing is different since once the original is prepaid, the debtor may enter into another contract, and the payoff should be minimized under this transaction (from one contract to another).

2.3 Previous Work

2.3.1 Structure-form

The measurement of prepayment incentive for option-based approach is endogenous. Many of the option-based approaches have been proposed, both in academic and practitioner sides. The termination behaviour is modelled as the optimal response of a rational debtor to the changes in some potential state variables, such as interest rate and house price. This type of model is closely related to value the early exercise feature of American options. The previous literature assumes that the debtors will follow an optimal call strategy. A rational debtor will compare the liability and outstanding principal to make decisions of immediate refinancing or postponing for an additional period, where the liability to the debtor and asset to the lender are not differentiated. In these papers, (see Dunn and McConnell ([13], [14]), Bernnan and Schwarze ([8]), Kau et al ([19], [20]) and relevant references contained therein), the debtor followed the behaviour that he or she would exercise his or her call option whenever the value of mortgage exceeded the remaining balance plus transaction costs, while Stanton ([36]) argued that this approach was not suitable when we considered structural changes in the economic environment.

The early work of valuing the default-free Government National Mortgage Association mortgage-backed pass-through securities was carried out by Dunn and McConnell ([13]). The model was based on a general equilibrium theory of the term structure of the interest rates. They adopted the contingent claim techniques to generate the price of the securities, and to avoid arbitrage opportunity. They constructed a PDE related to the value of a GNMA security, risk-free interest rate, and the probability of suboptimal prepayment, where Poisson-driven or jump process was considered to describe the suboptimal prepayments. Numerical solutions were presented by solving the PDE. Afterwards, Dunn and McConnell ([14]) continued the research and compared the price of GNMA mortgage-backed securities (denoted as MBS) with other types of fixed-bonds such as (1) nonamortizing, noncallable coupon bonds, (2) nonamortizing, callable coupon bonds, and (3) amortizing,noncallable coupon bonds. The comparisons provided the evidence that the impact of the call, amortization and prepayment features on GNMA securities. The results suggested that the features of callability would decrease the MBS price while the feature of amortization and prepayment had a positive effect.

Brennan and Schwartz have proposed a two-state variable model, including short rate and consol rate, to value the interest-dependent claims, i.e. default-free bonds and options, in the series of papers ([4], [5], [6], [7]). In [8], the authors priced GNMA securities through contrasting three different arbitrage-based models of the yield curve. The yield differentials were influenced by the interest-rate uncertainty and call policy.

Transaction costs were introduced by Dunn and Spatt ([15]), Timmis ([38]) and Johnston and Drunen ([17]). As the debtors may refinance as many times as they can in the future, refinancing costs will reduce the incentive of refinancing. The model proposed by Dunn and Spatt ([15]), was developed to value the debt contract with refinancing. In their assumption, the immediate benefit from refinancing was equalled to the refinancing costs and call premium at the refinancing point. The bound on the pricing of debt contracts was obtained, and the method could be applied even if the debtor would like to refinance recursively with transaction costs. In addition, Dunn and Spatt ([15]) indicated a new method to handle transaction cost. The transaction cost could be regraded as a refinancing option, which will be included in the agreement or contract. The direction is of vital importance since it has significant influence on the subsequent research. However, the main shortcoming is that the model implies all of the bahaviours of refinancing occur simultaneously in the same pool.

Chen and Ling ([9]) followed the previous research and developed a dynamic model of mortgage refinancing in a contingent claim for fixed-rate mortgage. With a binomial interest rate process, they have solved (1) the optimal mortgage refinancing strategy, (2) the value of the refinancing option, (3) the value of the mortgage liability to the debtor, and (4) the value of the contract, simultaneously. They assumed that a debtor would prepay the outstanding debt when the present value of interest rate savings exceeded the refinancing costs. Their model incorporated the possibility of recursive refinancing. IDF (interest rate differentials between the current market rate and contract rate) was first demonstrated in this paper, and the results of which contained the required minimum IDF for refinancing. The result showed that the IDF would increase with transaction costs, interest rate volatility and debtor's expected holding period.

Kau et al ([19]) incorporated possibility of default in valuing MBS, which occurred when the house value was less than the market value of the loan. Due to the fact that prepayment is dependent on the fluctuation of interest rate and default is concerned with the value of the house, the valuation of any asset is a function of time, house price and interest rate. The numerical results showed that responding of default to the economic environment was quite different from that of prepayment, and the marginal value of default was largely dependent on price volatilities. The work of Kau et al ([20]) extended on pricing of adjustable-rate mortgages and made comparison between these and fixed-rate mortgages with default.

Stanton ([36]) observed the drawbacks of reduced-form models, and the major one of which was that the prepayment model had low out-of-sample forecasting power. Stanton ([36]) incorporated both rational and exogenously determined prepayment strategies. He estimated heterogeneity in transaction cost faced by the debtors. Compared to Dunn and McConnell ([13], [14]), Dunn and Spatt ([15]), Timmis ([38]), Johnston and Drunen ([17]) et al, Stanton ([36]) assumed that the debtor would make decisions at discrete time. He acknowledged that some debtors would prepay even their coupon rate was below current rate, which meant these debtors failed to repay even at the optimal time. The model gave a simple model for rational prepayment, which was allowed to address the consequence of a structural shift in economic, such as seasonality.

Stanton and Wallace ([37]) developed the first contingent claims mortgage valuation algorithm of self-election, which allowed the debtors to choose the different fixed-rate loans with combinations of coupon rate and points, and an equilibrium model was proposed with transaction costs. Although some literature have investigated the similar problem before (see Yang ([47]), Leroy ([22])), they were unable to construct an equilibrium in multiple refinancing. The numerical solutions in the paper [37] demonstrated that, in determining the optimal menu of the mortgage contracts, the shape of yield curve, the transaction cost and the mobility of the debtors played an significant important role.

As the past option-based models focused on trying to predict future cash flows, Kalotay et al ([18]) concentrated on the market value of MBS. The reasons for the failure of past option-based models were as followings. The previous models either used Treasury or swap curves to model the behaviour refinancing. However, these curves could not accurately reflect the actual cost of funds, which led to the fact that the past option-based models were not able to explain and match market MBS prices. Instead, Kalotay et al ([18]) used two different yield curves, one for discounting mortgage cash flows and the other for MBS cash flows. By assuming that the sole purpose of refinancing was to save interest expense, they modelled the full spectrum of refinancing behaviour by a notion refinancing efficiency. They demonstrated that a rigorously constructed option-based model could accurately explain the market price and MBS were well priced when most debtors exercise their refinancing option near-optimally.

Nakagawa and Shouda ([29]) proposed a model which explained the heterogeneity of prepayments in the actual MBS market. The debtor's prepayment cost, which should be evaluated as the difference of the present values of the remaining mortgage between when the prepayment option was exercised or not exercised, was modelled as a stochastic process and the debtor's prepayment time was defined as when his or her prepayment cost process fell below zero. The conditional distribution of prepayment time in the loan pool given the debtors' filtration could be represented in terms of non-payment probability and the posterior density of loan pool risk.

Longstaff ([23]) studied the optimal recursive refinancing problem. They used the two-factor term structure model to describe interest rate fluctuation. The remarkable improvement was that the approach incorporated three factors on the optimal refinancing strategy: transaction cost, the probability of prepaying for exogenous reasons and the debtor's financial situation. Longstaff borrowed method in Longstaff and Schwartz ([24]) to compute the prices of the prepayment options and the mortgage contracts. The results illustrated that it was optimal to delay prepayment for the debtor beyond the point when compared to the conventional models.

2.3.2 Reduced-form

Considering the fact that the debtors prepay their loans even the prevailing refinancing rate exceeds their initial contract rate, and other debtors do not prepay when the initial contract rate exceeds the prevailing rate, Schwartz and Torous ([31]) have modelled the factors such as economic, demographic and geographic elements, which would influence the debtor's decision by statistical estimation. Schwartz and Torous ([31]) incorporated an empirical prepayment function into a two-factor default-free interest-dependent claim and led to a partial differential equation for the value of mortgage contract. One significance is that in this research, it is recognized that at each time, there exists a probability of prepaying, that the random time when a debtors prepays could be described as a hazard rate model. They provided a complete model to value the MBS. The later work of Schwartz and Torous ([32]) was the first to introduce the possibility of default and investigate the interaction of prepayment and default decisions for valuing MBS. With transaction costs, the conditional probability of prepayment or default was given by the function of prepayment or default, separately. In an arbitrage free market, the value of the mortgage or mortgage pass-through satisfied the second-order partial differential equation. Although some of the reduced-form models can be quite complicated, it is straightforward to use the Monte Carlo simulation. In 1993, Schwartz and Torous ([33]) took advantage of Poisson regression to estimate the parameters of a proportional hazards model instead of likelihood method, which was more efficient to obtain the result.

As in reality, the debtors do not have perfect information about future interest rate movement and there exists transaction, it is not appropriate to apply the reduced-form models with deterministic term structures. Deng ([11]) incorporated a binomial mean-reverting interest rate model into the hazard framework, and analyzed the residential mortgage prepayment and default risk by a unified economic model of contingent claims. The authors considered prepayment risk and default risk as interdependent competing risk and estimated them jointly by a semi-parametric estimation approach. The results showed that the uncertainty of interest rate movement and liquidity constraints affected both on predicting mortgage prepayment and default behaviour. Deng et al ([12])

extended the unified economic model to analyze the heterogeneity among debtors, which was quite important in accounting for their prepayment and default behaviour.

2.4 Interest Rate Models

2.4.1 Merton model

Merton ([27]) proposed the following simplest stochastic process for the dynamic of interest rate

$$r_t = r_0 + ut + \sigma W_t,$$

where the u and σ are constants, and W_t is the standard Brownian process. As r_t follows the normal distribution with mean $r_0 + ut$ and variance $\sigma^2 t$, the moment generation function of r_t is

$$M_{r_t}(z) = e^{(r_0+ut)z+\frac{1}{2}\sigma^2 z^2 t}.$$

The first and second moments of r_t are unbounded, which allow the interest rate r_t to be infinity. In a sense, the model lacks stability and cannot be applicable to all the conditions.

2.4.2 Vacicek model

The Vasicek model was introduced by Vasicek in 1977 ([39]). This model can be used to interest rate derivative valuation and also adapted to credit market. Vasicek Model is an Ornstein-Uhlenbeck stochastic process given by

$$dr_t = k(\theta - r_t)\,dt + \sigma dW_t,$$

where reversion rate k, long-term mean level θ, volatility σ are positive constants, and W_t is the standard Brownian process. Vasicek model was the first one to capture mean reversion, which defined an elastic random walk around the trend.

- θ: 'long term mean level'. The long run equilibrium value towards which the interest rate goes back, which means all future trajectories of r_t will evolve around a mean level θ in the long run.

- k: 'speed of reversion'. It gives the adjustment of speed and has to be positive in order to maintain stability around for the long-term value.

- σ: 'instantaneous volatility'. It determines the volatility of the interest rate, and higher σ implies more randomness.

- $k(\theta - r_t)\,dt$: 'drift term'. The drift factor that describes the expected change in the interest rate at that particular time.

- $\frac{\sigma^2}{2k}$: 'long term variance'. All future trajectories of r_t will revert around the long term mean with such variance after a long time.

When r_t goes under θ, the drift term $k(\theta - r_t)$ becomes positive, generating a tendency for the interest rate to move upwards, and vice versa.

Vasicek model was the first one to capture mean reversion property of the interest rate. Unlike stock price, the model assumes interest rate moves within a limited range, which shows tendency of the interest movement will finally revert to a long run value. However, the main drawback of Vasicek model is that the short term interest rate can become negative, which is not acceptable at the economic point-of-view.

Vasicek model yields an explicit formula

$$r_t = \theta + (r_0 - \theta)e^{-kt} + \sigma e^{-kt} \int_0^t e^{ku} dW_u,$$

with

$$\begin{aligned} \mathrm{E}[r_t] &= \theta + (r_s - \theta)e^{-kt} \\ \mathrm{V}ar[r_t] &= \sigma^2 e^{-2kt} \mathrm{E}\Big[\Big(\int_0^t e^{ku} dW_u\Big)^2\Big] = \sigma^2 e^{-2kt} \mathrm{E}\Big[\int_0^t e^{2ku} du\Big] = \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right). \end{aligned}$$

One can see that r_t is a Gaussian random variable. This follows from the definition of the stochastic integral term $\sigma e^{-kt} \int_0^t e^{ku} dW_u$, which is $\lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} \sigma e^{-k(t-u_i)} \left(W_{u_{i+1}} - W_{u_i}\right)$. As the increment is $W_{u_{i+1}} - W_{u_i} \sim N(0, u_{i+1} - u_i)$, $\int_0^t e^{2ku} dW_u$ is Gaussian.

As r_t follows normal distribution with mean of $\theta + (r_0 - \theta)e^{-kt}$ and variance of $\frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)$, the moment generating function of r_t is

$$M_{r_t}(z) = e^{\left(\theta + (r_0 - \theta)e^{-kt}\right)z + \frac{\sigma^2}{4k}\left(1 - e^{-2kt}\right)z^2}.$$

Compare to Merton Model, Vasicek Model avoids the infinite interest rate. However, the main disadvantage of Vasicek Model is that interest can be negative. When $t \to \infty$, we have

$$\begin{cases} \lim_{t \to \infty} \mathrm{E}[r_t] &= \theta \\ \lim_{t \to \infty} \mathrm{V}ar[r_t] &= \frac{\sigma^2}{2k}. \end{cases}$$

As the explicit formula is given, one can obtain the zero-coupon bond price in the following way in the paper of Mamon ([26]). Using the risk-neutral valuation framework, the price of a zero-coupon bond with maturity T at time t is

$$B(t,T) = \mathrm{E}\Big[e^{-\int_t^T r_u du} | F_t\Big].$$

We let $X_t = r_t - \theta$, as X_t is the solution of the Ornstein-Uhlenbeck equation, we have

$$dX_t = -kX_t dt + \sigma dW_t,$$

with the initial value

$$X_0 = r_0 - \theta.$$

Applying Ito's lemma formula, X_t is given by

$$X_t = e^{-kt} X_0 + \sigma e^{-kt} \int_0^t e^{ks} dW_s,$$

with

$$\begin{aligned}
\mathrm{E}[X_t] =& e^{-kt} X_0 \\
\mathrm{Cov}[X_t, X_u] =& \sigma^2 e^{-k(u+t)} \mathrm{E}\left[\int_0^t e^{av} dW_v \int_0^u e^{av} dW_v \right]. \\
=& \frac{\sigma^2}{2k} e^{-k(u+t)} \left(e^{2k(u \wedge t)} - 1 \right)
\end{aligned}$$

As X_u is a Gaussian process with continuous sample paths, then $\int_0^t X(u)du$ is also Gaussian, with

$$\begin{aligned}
\mathrm{E}\left[\int_0^t X_u du \right] =& \int_0^t \mathrm{E}[X_u] du = \frac{X_0}{k} \left(1 - e^{-kt} \right) \\
\mathrm{V}ar\left[\int_0^t X_u du \right] =& \mathrm{Cov}\left[\int_0^t X_u du, \int_0^t X_v dv \right] \\
=& \int_0^t \int_0^t \mathrm{Cov}[X_u, X_v] du dv \\
=& \frac{\sigma^2}{2k^3} \left[-e^{-2kt} + 4e^{-kt} + 2kt - 3 \right].
\end{aligned}$$

Since $r_u = X_u + \theta$, we have

$$\begin{aligned}
\mathrm{E}\left[\int_t^T r_u du \right] =& \mathrm{E}\left[\int_t^T (X_u + \theta) du \right] \\
=& - \frac{r_t - \theta}{k} \left(1 - e^{-k(T-t)} \right) + \theta(T - t) \\
\mathrm{V}ar\left[\int_t^T r_u du \right] =& \mathrm{V}ar\left[\int_t^T X_u du \right] \\
=& \frac{\sigma^2}{2k^3} \left[-e^{-2k(T-t)} + 4e^{-k(T-t)} + 2k(T-t) - 3 \right].
\end{aligned}$$

Thus, the value of the zero-coupon bond price can be described as

$$\begin{aligned}
B(t,T) = \mathrm{E}\left[e^{-\int_t^T r_u du} | F_t \right] =& \mathrm{E}\left[e^{-\int_t^T r_u du} | r_t \right] \\
=& e^{\left[-\int_t^T r_u du \right] + \frac{1}{2} \mathrm{V}ar\left[-\int_t^T r_u du \right]} \\
=& A_1(t,T) e^{-A_2(t,T) r_t},
\end{aligned}$$

where

$$\begin{aligned}
A_1(t,T) =& \exp\left(\left(\theta - \frac{\sigma^2}{2k} \right) [A_2(t,T) - (T-t)] - \frac{\sigma^2 A_2^2(t,T)}{4k} \right) \\
A_2(t,T) =& \frac{1 - e^{-(T-t)k}}{k}.
\end{aligned}$$

2.4.3 CIR model

The CIR short term interest rate process, first proposed by Cox et al ([10]), is a mathematical model describing the evolution of interest rate. The model specifies that under the risk-neutral measure Q, the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t. \tag{2.1}$$

CIR model is one of the most well-known and widely used models for interest rate and the pricing of interest rate derivatives, by which many books and books have adopted to capture the term structure of interest rate (see Shreve ([35]), Dunn and McConnell ([13], [14]), Sharp ([34]), Miranda-Mendoza ([28]) et al). It is composed of one deterministic term and one random term. The deterministic term (also 'the drift term') is chosen to produce the so called 'mean-reverting' property, which means that if the interest rate is larger than the long-term mean, the drift term will be negative so that the interest rate will be pulled down in the direction of the long-term mean. However, if the interest rate is smaller than the long-term mean, the drift term will be positive so that the interest rate will be pulled up in the direction of the long-term mean. And the random term is to model the volatility caused by unpredictable factors. In (2.1), k is the reversion rate, which refers to the speed measuring how fast the process will be reverted back to the mean once it evolves away from the mean, while θ is long$-$term mean interest rate and σ is the standard deviation, all of which are positive constants. When we add one condition $2k\theta > \sigma^2$, the interest rate is always positive, otherwise the interest rate can reach zero. The volatility term σ is multiplied with the term $\sqrt{r_t}$, which eliminates the probability of negative interest rates compared to the Vasicek model. The main reason to adopt CIR model to generate the mortgage rate is that it avoids the negative rates, and corresponds to empirical observations that higher interest rates are associated with higher volatility, which guarantee that our simulated mortgage rate is more realistic. The probability density function of r_s, conditional on r_v, where $s > v$, is given by Cox et al ([10])

$$f_{r_s|r_v}(x) = ae^{-br_v - ax}\left(\frac{ax}{br_v}\right)^{\frac{c}{2}} I_c\left(2\sqrt{abr_v x}\right), \quad x > 0 \tag{2.2}$$

where

$$a = \frac{2k}{\sigma^2\left(1 - e^{-k(s-v)}\right)}, \quad b = ae^{-k(s-v)}, \quad c = \frac{2k\theta}{\sigma^2} - 1,$$

and $I_c(y)$ is the Modified Bessel's function of the first kind of order c, which is

$$I_c(y) = \sum_{m=0}^{\infty} \frac{\left(\frac{y}{2}\right)^{2m+c}}{m!\Gamma(m+c+1)}.$$

Based on (2.2), we can calculate the moment generating function as

$$
\begin{aligned}
M_{r_s|r_v}(t) = \mathrm{E}\left[e^{tr_s}|r_v\right] &= \int_0^\infty e^{tx} a e^{-br_v-ax}\left(\frac{ax}{br_v}\right)^{\frac{c}{2}} I_c\left(2\sqrt{abr_v x}\right) \\
&= \int_0^\infty a e^{-br_v}\left(\frac{a}{br_v}\right)^{\frac{c}{2}} \sum_{m=0}^{\infty} \frac{\left(\sqrt{abr_v}\right)^{2m+c}}{m!\Gamma(m+c+1)} e^{-ax} x^{\frac{c}{2}} x^{\frac{2m+c}{2}} e^{tx} dx \\
&= \int_0^\infty a e^{-br_v}\left(\frac{a}{br_v}\right)^{\frac{c}{2}} \sum_{m=0}^{\infty} \frac{\left(\sqrt{abr_v}\right)^{2m+c}}{m!\Gamma(m+c+1)} e^{-(a-t)x} x^{m+c} dx \\
&= a e^{-br_v}\left(\frac{a}{br_v}\right)^{\frac{c}{2}} \sum_{m=0}^{\infty} \frac{\left(\sqrt{abr_v}\right)^{2m+c}}{m!\Gamma(m+c+1)} \Gamma(m+c+1)\left(\frac{1}{a-t}\right)^{m+c+1} \\
&= \sum_{m=0}^{\infty} e^{-br_v} br_v^m a^{m+c+1} \frac{1}{m!}\left(\frac{1}{a-t}\right)^{m+c+1} \\
&= \left(\frac{a}{a-t}\right)^{c+1} \sum_{m=0}^{\infty} \frac{e^{-br_v}\left(\frac{abr_v}{a-t}\right)^m}{m!} \\
&= \left(\frac{a}{a-t}\right)^{c+1} e^{\frac{br_v t}{a-t}},
\end{aligned}
$$

and specifically, the moment generating function of r_s, conditional on r_0 (ie, $v = 0$), is

$$
M_{r_s}(t) = \left(\frac{a}{a-t}\right)^{c+1} e^{\frac{br_0 t}{a-t}},
$$

where

$$
a = \frac{2k}{\sigma^2\left(1-e^{-ks}\right)}, \quad b = ae^{-ks}, \quad c = \frac{2k\theta}{\sigma^2} - 1.
$$

In addition, the zero-coupon bond price based on CIR model is given in the following section.

2.5 Term Structure of Interest Rate

For more details, the reader may refer to Gibson et al [16]).

2.5.1 Definitions

In the rational financial market, a lender will never lend money for free. As the value of money is always higher today than future, the lender will charge for borrowed money as the compensation for the loss of the future opportunities one could miss out for the borrowed money.

The term-structure of interest rates refers to different interest rates that exist over different term-to-maturity loans. As we only consider zero-coupon bonds, the yield curve is the same as the term-structure of interest rates. We denote $B(t,T)$ as the discount bond price of zero-coupon bond from current time t to the maturity time T. At time t, the yield to maturity $R(t,T)$ of the discount bond $B(t,T)$ follows

$$B(t,T)e^{(T-t)R(t,T)} = 1,$$

and thus, $R(t,T)$ is represented as,

$$R(t,T) = -\frac{\ln [B(t,T)]}{T-t}, \tag{2.3}$$

where $R(t,T)$ is the continuously compounded interest rate. When we fix t, one can see that the yield curve is determined by T.

We define $r(t)$ as the spot rate at time t, then

$$\begin{aligned} r(t) &= \lim_{T \to t} R(t,T) = - \lim_{\Delta t \to 0} R(t, t+\Delta t) \\ &= - \frac{\ln [B(t,t+\Delta t)]}{\Delta t}. \end{aligned}$$

As $B(t,t) = 1$, we have

$$r(t) = -\frac{d \ln [B(t,T)]}{dT}|_{T=t}.$$

We denote $f(t,T_1,T_2)$ as forward rate, which can be agreed on the current time t for a risk-free loan from T_1 to T_2. Similarly, the instantaneous forward rate is

$$f(t,T) = -\frac{d \ln [B(t,T)]}{dT}, \tag{2.4}$$

which gives

$$B(t,T) = e^{-\int_t^T f(t,u)du}.$$

Note that in our thesis, we only focus on the short rate model, which is given by the following stochastic differential equation

$$dr_t = u(t,r_t)dr_t + \sigma(t,r_t)dW_t,$$

which implies r_t is a Markov process, and the zero-coupon bond price given by

$$B(t,T) = \mathrm{E}\Big[e^{-\int_t^T r_s ds}|F_t\Big].$$

2.5.2 The theories

The Expectation Theory: This theory assumes that the implied forward rates are unbiased estimates of the future prevailing spot rates. That is, the realized difference between the actual spot interest rate and any previous periods forward interest rate is, on average, zero. The key assumption behind this theory is that the rate of return on a bond maturating at time T should be equal to the geometric average of the short-term rate from t to T.

The Market Segmentation Theory: The key assumption of this theory is that bonds of different maturities are not substitutes at all. It implies markets are completely segmented, and interest rate at each maturity are determined separately. As bonds of shorter holding periods have lower inflation and interest rate risks, segmented market theory predicts that yield on longer bonds will generally be higher, which explains why the yield curve is usually upward sloping.

The Liquidity-Preference Theory: The liquidity premium theory views bonds of different maturities as substitutes, but not perfect substitutes. Investors prefer short rather than long bonds because they are free of inflation and interest rate risks. This implies that the prices of longer-term bonds tend to be more volatile than the prices of short-term bonds, resulting in a higher expected return, or risk premium, to offset the higher risk.

2.5.3 Interest rate derivative pricing: PDE approach

Recall that we denote $B(t,T)$ as the discount bond price of zero-coupon bond. In the risk-neutral world, one can define $B(t,T)$ as (see Shreve ([35]))

$$B(t,T) = \mathrm{E}\Big[e^{-\int_t^T r_s ds}|F_t\Big].$$

We assume in the general case, the interest rate follows

$$dr_t = u(t,r_t)dr_t + \sigma(t,r_t)dW_t,$$

where $u(t,r_t)$ and $\sigma(t,r_t)$ are functions related to t and r_t.

We consider the short term rate is the single factor deriving the term structure (see Gibson et al [16]). Thus, we can derive a PDE for valuation $B(t,T)$, whose value is a function of interest rate r_t, time t and maturity date T. Applying Ito lemma ([35]) to the function $B(t,T)$, we obtain that

$$\begin{aligned} dB =& \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial r_t}dr_t + \frac{\partial^2 B}{\partial r_t^2}(dr_t)^2 \\ =& a(t,r_t)dt + b(t,r_t)dW_t, \end{aligned}$$

with

$$\begin{aligned} a(t,r_t) =& a = \frac{\partial B}{\partial t} + u(t,r_t)\frac{\partial B}{\partial r_t} + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 B}{\partial r_t^2} \\ b(t,r_t) =& b = \sigma(t,r_t)\frac{\partial B}{\partial r_t}. \end{aligned} \tag{2.5}$$

Now construct a portfolio Π, which consists of long one asset B_1 and short Δ of B_2. Thus

$$\Pi = B_1 - \Delta B_2.$$

The change in the portfolio over dt is

$$
\begin{aligned}
d\Pi =& dB_1 - \Delta dB_2 \\
=& (a_1 - \Delta a_2)dt + (b_1 - \Delta b_2)dW_t.
\end{aligned}
$$

To eliminate the risk of the portfolio, we choose

$$\Delta = \frac{b_1}{b_2}.$$

As the return on an amount Π invested in riskless assets would see a growth of $r\Pi dt$ in a time dt.

$$r\Pi dt = (a_1 - \frac{b_1}{b_2}a_2)dt,$$

which gives

$$\frac{a_1 - rB_1}{b_1} = \frac{a_2 - rB_2}{b_2}. \tag{2.6}$$

As the (2.6) holds for any pair of B_1 and B_2, the ratio of $\frac{a-rB}{b}$ needs to be only concerned with r and t. We denote the market premium $\lambda(r,t) = \frac{a-rB}{b}$. In compatible with the no-arbitrage requirement, we can assume that $\lambda(r,t) = 0$, which gives

$$a = rB, \tag{2.7}$$

where a is defined in equation (2.5). Substituting equation (2.5) into (2.7), we have

$$\frac{\partial B}{\partial t} + u(t, r_t)\frac{\partial B}{\partial r_t} + \frac{1}{2}\sigma^2(t, r_t)\frac{\partial^2 B}{\partial r_t^2} - r_t B = 0, \tag{2.8}$$

We can guess the solution with the form of

$$B(t,T) = A_1(t,T)e^{-A_2(t,T)r_t}.$$

Thus

$$
\begin{aligned}
\frac{\partial B}{\partial t} =& A_1'(t,T)e^{-A_2(t,T)r_t} - A_1(t,T)A_2'(t,T)r_t e^{-A_2(t,T)r_t} \\
\frac{\partial B}{\partial r_t} =& -A_1(t,T)A_2(t,T)e^{-A_2(t,T)r_t} \\
\frac{\partial^2 B}{\partial r_t^2} =& A_1(t,T)A_2^2(t,T)e^{-A_2(t,T)r_t},
\end{aligned}
$$

where

$$A_1'(s,t) = \frac{dA_1(t,T)}{dt}, \quad A_2'(s,t) = \frac{dA_2(t,T)}{dt}.$$

We adopt Merton's Model as an example, where $u(t, r_t) = u$ and $\sigma(t, r_t) = \sigma$, and plug $\frac{\partial B}{\partial t}$, $\frac{\partial B}{\partial r_t}$, $\frac{\partial^2 B}{\partial r_t^2}$ in (2.8) gives

$$A_1'(s,t) - uA_1(s,t)A_2(s,t) + \frac{1}{2}\sigma^2 A_1(s,t)A_2^2(s,t) - \left[A_1(s,t)A_2'(s,t) + A_1(s,t)\right] r_t = 0. \tag{2.9}$$

As the (2.9) holds for all r_t, we can figure out that

$$\begin{cases} A_1'(s,t) - uA_1(s,t)A_2(s,t) + \frac{1}{2}\sigma^2 A_1(s,t)A_2^2(s,t) = 0 \\ A_1(s,t)A_2'(s,t) + A_1(s,t) = 0. \end{cases} \tag{2.10}$$

With the boundary conditions $A_1(T,T) = 1$ and $A_2(T,T) = 0$, we can obtain

$$\begin{aligned} A_1(t,T) =& \exp\left(-\frac{u(T-t)^2}{2} + \frac{\sigma^2(T-t)^3}{6}\right) \\ A_2(t,T) =& T - t. \end{aligned}$$

Similar calculation can be worked out with Vasicek Model and CIR Model, and the results are as followings: For Vasicek Model, we have

$$\begin{aligned} A_1(t,T) =& \exp\left(\left(\theta - \frac{\sigma^2}{2k^2}\right)\left[A_2(t,T) - (T-t)\right] - \frac{\sigma^2 A_2^2(t,T)}{4k}\right) \\ A_2(t,T) =& \frac{1 - e^{-(T-t)k}}{k}. \end{aligned}$$

For CIR Model, we have

$$\begin{aligned} A_1(t,T) =& \left(\frac{2\omega e^{\frac{(k+\omega)(T-t)}{2}}}{2\omega + (k+\omega)\left[e^{(T-t)\omega} - 1\right]}\right)^{\frac{2k\theta}{\sigma^2}} \\ A_2(t,T) =& \frac{2\left[e^{(T-t)\omega} - 1\right]}{2\omega + (k+\omega)\left[e^{(T-t)\omega} - 1\right]} \\ \omega =& \sqrt{k^2 + 2\sigma^2}. \end{aligned}$$

2.5.4 Feynman-Kac formula

The Feynman-Kac formula (see [41]) establishes a link between parabolic partial differential equations (PDEs) and stochastic processes.

Theorem 2.5.1. *Let X_t be a stochastic process satisfying*

$$dX_t = u(X_t, t)dt + \sigma(X_t, t)dW_t.$$

Let $F(X_t, t)$ be the price at time of t of any derived security in the economy maturing at T , with the maturity price

$$F(X_T, T) = g(X_T),$$

and one can derive a PDE

$$\frac{\partial F}{\partial t} + u(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} - V(t,x)F = 0,$$

with the boundary condition

$$F(T,x) = g(x) \quad \textit{for all } x.$$

Then the Feynman $-$ Kac formula tells us that the solution can be written as a conditional expectation

$$F(t,x) = \mathrm{E}\Big[g(x_T)e^{-\int_t^T V(u,X_u)du}|X_t = x\Big].$$

Proof. We assume $F(t,x)$ is the solution of the PDE, and we construct

$$Y(s) = e^{-\int_t^s V(u,X_u)du}F(s,X_s).$$

Apply Ito's Lemma, we have

$$\begin{aligned} dY(s) =& F(s,X_s)de^{-\int_t^s V(u,X_u)du} + e^{-\int_t^s V(u,X_u)du}dF(s,X_s) + de^{-\int_t^s V(u,X_u)du}dF(s,X_s) \\ =& F(s,X_s)de^{-\int_t^s V(u,X_u)du} + e^{-\int_t^s V(u,X_u)du}dF(s,X_s) \\ =& e^{-\int_t^s V(u,X_u)du}\left[-V(s,X_s)F(s,X_s) + dF(s,X_s)\right]. \end{aligned}$$

As

$$\begin{aligned} dF(s,X_s) =& \frac{\partial F}{\partial s}ds + \frac{\partial F}{\partial x}dx + \frac{\partial^2 F}{\partial x^2}(dx)^2 \\ =& \frac{\partial F}{\partial t} + u(s,X_s)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2} + \sigma(s,X_s)\frac{\partial F}{\partial x}dW, \end{aligned}$$

$dY(s)$ can be continued as

$$\begin{aligned} =& e^{-\int_t^s V(u,X_u)du}\left[-V(s,X_s)F(s,X_s) + \frac{\partial F}{\partial s} + u(s,X_s)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2} + \sigma(s,X_s)\frac{\partial F}{\partial x}dW\right] \\ =& e^{-\int_t^s V(u,X_u)du}\sigma(s,X_s)\frac{\partial F}{\partial x}dW. \end{aligned}$$

Integrating this equation from t to T, we can obtain that

$$Y(T) - Y(t) = \int_t^T e^{-\int_t^s V(u,X_u)du}\sigma(s,X_s)\frac{\partial F}{\partial x}dW.$$

Taking the expectation, conditioned on $X_t = x$ of both sides implies

$$\mathrm{E}[Y(T)|X_t = x] = \mathrm{E}[Y(t)|X_t = x] = F(t,x).$$

Thus

$$\begin{aligned} F(t,x) =& \mathrm{E}\Big[F(T,X_T)e^{-\int_t^T V(u,X_u)du}\Big] \\ =& \mathrm{E}\Big[g(x_T)e^{-\int_t^T V(u,X_u)du}|X_t = x\Big]. \end{aligned}$$

□

One can see that when the payoff $g(x_T) = 1$, the formula can be adopted to the calculation of bond price.

Chapter 3

Modelling of Refinancing

3.1 Business Economic Assumptions

As market increasingly diversifies, the mortgage contract itself becomes rather complicated in real industry, the documentation of which concerns not only financial and business consultants, but also commercial lawyers and regulatory compliance, etc. This said, it is reasonable for us to summarise common contract specifics and economic environment in which the mortgage deals are cultivated.

1. With the continuous payment, one refinancing is granted throughout the whole during of the original contract. The transaction fee is charged as the percentage of the profit gained by refinancing. If the profit is M_s, the lender may charge the transaction fee as βM_s, with $\beta \in (0, 1)$. In addition, the life of the contract will not be affected by refinancing.

2. No prepayment or default will be considered in this thesis.

3. The market is complete, and both the lender and the debtor have equal access to the market information.

4. The debtor does not have a sizable enough amount of fund to make early payment.

Among these assumptions, 1-2 are contract clauses or interpretations of these clauses; and 3-4 are market and economic environment assumptions. In particular, the assumption 3 guarantees the method and solutions contained in this thesis are arbitrage free.

3.2 Model Setting for Mortgage Refinancing

1. r_t: market interest rate at time t, we define $e^{-\int_0^s r_t dt}$ as the discount process to time s.

2. T: the duration of mortgage contract, in the unit of years, $t \in [0, T]$.

3. $c(t) = c_t$: mortgage rate contracted at t, for the time interval $[t, T]$. c_t is a deterministic function of r_t.

4. $P(t)$: Consider a bank loan of amount $P(0)$ at $t = 0$ for the duration of T. $P(t)$ is the principal balance at time t, which implies if the debtor wants to pay off the debt at time t, he or she needs to repay $P(t)$. Therefore, $P(t)$ equals $\frac{P(0)}{1-e^{-c_0 T}}\left(1 - e^{-c_0(T-t)}\right)$ and at maturity date $t = T$, $P(T) = 0$.

5. m_t: rate of payment per unit amount of loan determined at t for the duration of $[t, T]$. The payment rate per unit amount and the mortgage rate satisfy

$$\int_t^T e^{-c_t(s-t)} m_t ds = 1,$$

which gives

$$-m_t \frac{1}{c_t} e^{-c_t(s-t)} \Big|_t^T = \frac{m_t}{c_t}\left(1 - e^{-c_t(T-t)}\right) = 1,$$

or equivalently

$$m_t = \frac{c_t}{1 - e^{-c_t(T-t)}}. \tag{3.1}$$

6. We consider a portfolio V consisting of a loan of $P(0)$ at time $t = 0$ for the duration of T years with the mortgage rate c_0 and a refinancing agreement to be exercised at time $s \in (0, T)$, if the mortgage rate c_s at time s satisfies $c_s < c_0$. Initially, the debtor would undertake the continuous payment rate of $m_0 P(0)$ with the mortgage rate c_0. At time $t = s$, if refinancing is exercised leads to a new payment rate of $m_s P(s)$ with the mortgage rate c_s. If M_s is the value of this portfolio at time $t = 0$, with the market interest rate r_s at time s, we have

$$M_s = \begin{cases} \displaystyle\int_s^T \left[m_0 P(0) - m_s P(s)\right] e^{-\int_0^t r_v dv} dt & c_s < c_0 \\ 0, \qquad c_s \geq c_0, \end{cases} \tag{3.2}$$

where

$$m_0 P(0) - m_s P(s) = P(0)\left[\frac{c_0}{1 - e^{-c_0 T}} - \frac{c_s\left[1 - e^{-c_0(T-s)}\right]}{\left[1 - e^{-c_0 T}\right]\left[1 - e^{-c_s(T-s)}\right]}\right].$$

M_s can be also viewed as the total discounted profit of refinancing at time s. As described, the lender may charge βM_s as the transaction fee. Thus, the profit gained by the debtor is $(1-\beta)M_s$. A natural question is to find the optimal time which maximizes the utility of the profit and the risk. Since M_s is a stochastic process, we may consider its expectation $\mathrm{E}[M_s]$ and its variance $\mathrm{V}ar[M_s]$ as key factors. If U: $\mathbb{R}^2 \to \mathbb{R}$ is such a utility function, our problem is equivalent to

$$U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{\mathrm{V}ar[M_s]}}\right), \tag{3.3}$$

where

$$\mathrm{E}[M_s] = \int_s^T \mathrm{E}\Big[[m_0 P(0) - m_s P(s)]\, e^{-\int_0^t r_v dv}\Big] dt, \tag{3.4}$$

and

$$\mathrm{V}ar[M_s] = \mathrm{E}\left[[m_0 P(0) - m_s P(s)]^2 \left(\int_s^T e^{-\int_0^t r_v dv} dt\right)^2\right] - (\mathrm{E}[M_s])^2. \tag{3.5}$$

In general, the unconstrained maximization problem $U(x,y)$ will be obtained by setting $U_x = 0$ and $U_y = 0$, with the second-order conditions $U_{xx} < 0$, $U_{yy} < 0$ and $\begin{vmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{vmatrix} < 0$. Thus, for a utility function $U(x,y)$, we can see that $\begin{vmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{vmatrix}$ is a negative-definite matrix.

We let $x(s) = \mathrm{E}[M_s]$ and $y(s) = \frac{1}{\sqrt{\mathrm{V}ar[M_s]}}$ in (3.3), thus, the optimal point will be obtained by

$$\frac{d}{ds} U\left(x(s), y(s)\right) = U_x x'(s) + U_y y'(s) = 0, \tag{3.6}$$

because

$$\begin{aligned}
&\frac{d^2}{ds^2} U\left(x(s), y(s)\right) \\
=&U_{xx}\left[x'(s)\right]^2 + 2U_{xy}x'(s)y'(s) + U_x x''(s) + U_{yy}\left[y'(s)\right]^2 + U_y y''(s) \\
=&\begin{pmatrix} x'(s), y'(s) \end{pmatrix} \begin{pmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{pmatrix} \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} + \begin{pmatrix} U_x, U_y \end{pmatrix} \begin{pmatrix} x''(s) \\ y''(s) \end{pmatrix} \\
<&0,
\end{aligned}$$

thus, the maximum value of $U\left(x(s), y(s)\right)$ will occur at s satisfying (3.6).

In particular, The utility function can be described by the Cobb-Douglas model (see [40]), where

$$U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{\mathrm{V}ar[M_s]}}\right) = (\mathrm{E}[M_s])^{\rho} \frac{1}{\left(\sqrt{\mathrm{V}ar[M_s]}\right)^{1-\rho}}, \tag{3.7}$$

where $\rho \in (0,1)$.

Without loss of generality, we assume $\mathrm{E}[M_s]$ and $\mathrm{V}ar[M_s]$ are continuous, positive and differentiable. The maximum value of $U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{\mathrm{V}ar[M_s]}}\right)$ will occur at s satisfying $\frac{dU}{ds} = 0$, implying

$$\rho \frac{d}{ds} \ln\left(\mathrm{E}[M_s]\right) = (1-\rho)\frac{d}{ds} \ln\left(\sqrt{\mathrm{V}ar[M_s]}\right). \tag{3.8}$$

With $\rho = 1$, the maximum value of $U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{Var[M_s]}}\right)$ will occur at s satisfying $\frac{d\mathrm{E}[M_s]}{ds} = 0$.

We assume the mortgage rate c_t is a function of r_t with $c_t \geq r_t$, and the market interest rate r_t satisfies the following SDE of

$$dr_t = u(t, r_t)dt + \sigma(t, r_t)dW_t,$$

where $u(t, r_t)$ is the drift coefficient, $\sigma(t, r_t)$ is the diffusion coefficient, and W_t is the standard Brownian motion.

The expectation of M_s can be represented as

$$\begin{aligned}
\mathrm{E}[M_s] &= \int_s^T \mathrm{E}\Big[[m_0P(0) - m_sP(s)]\, e^{-\int_0^t r_v dv}\Big]dt \\
&= \int_s^T \mathrm{E}\Big[\mathrm{E}\Big[[m_0P(0) - m_sP(s)]\, e^{-\int_0^t r_v dv}|r_s\Big]\Big]dt \\
&= \int_s^T \mathrm{E}\Big[[m_0P(0) - m_sP(s)]\, e^{-\int_0^s r_v dv}\mathrm{E}\Big[e^{-\int_s^t r_v dv}|r_s\Big]\Big]dt \\
&= \int_s^T \mathrm{E}\Big[[m_0P(0) - m_sP(s)]\, e^{-\int_0^s r_v dv}B(s,t)\Big]dt, \qquad (3.9)
\end{aligned}$$

where $B(s,t) = \mathrm{E}\Big[e^{-\int_s^t r_v dv}|r_s\Big]$ is zero-coupon discounted bond price with maturity t with explicit formula

$$B(s,t) = A_1(s,t)e^{-A_2(s,t)r_s}.$$

And the formulae of $A_1(s,t)$ and $A_2(s,t)$ will depend on the stochastic interest rate process we adopted.

We can rewrite (3.9) as

$$\begin{aligned}
\mathrm{E}[M_s] =&\mathrm{E}\left[[m_0P(0) - m_sP(s)]\, e^{-\int_0^s r_v dv} \int_s^T B(s,t)dt\right] \\
=&\mathrm{E}\left[[m_0P(0) - m_sP(s)]\, e^{-\int_0^s r_u du} \lim_{||\Pi|| \longrightarrow 0} \sum_{i=0}^{n} B\left(s, \frac{i}{n}(T-s)\right) \frac{T-s}{n}\right].
\end{aligned}$$

We construct a portfolio $B_s = \lim_{||\Pi|| \longrightarrow 0} \sum_{i=0}^{n} B\left(s, \frac{i}{n}(T-s)\right) \frac{T-s}{n}$, and the payment rate of the portfolio after refinance at time s denotes as $R_s = \frac{c_s\left[1-e^{-c_0(T-s)}\right]}{\left[1-e^{-c_0T}\right]\left[1-e^{-c_s(T-s)}\right]}$. Thus, we have

$$\mathrm{E}[M_s] =\mathrm{E}\Big[P(0)\left(R_0 - R_s\right)B_s e^{-\int_0^s r_v dv}\Big].$$

We may think the debtor holds a payment option. If the mortgage rate at time s, c_s, is lower than the contractual rate, c_0, which implies $R_s < R_0$, the debtor would like to exercise the option and new payment becomes $P(0)R_sB_s$, making a profit of $P(0)B_s(R_0 - R_s)$. Otherwise, the debtor will discard the option and keep the original contract.

Lemma 3.2.1. *If the mortgage rate $c_s < c_0$, then $R_s < R_0$ when $s \in (0,T)$.*

Proof. Since

$$\frac{R_s}{R_0} = \frac{c_s}{c_0}\frac{1-e^{-c_0(T-s)}}{1-e^{-c_s(T-s)}},$$

we let $f(x) = \frac{x}{1-e^{-x(T-s)}}$ with $x > 0$, thus, the first derivative of $f(x)$ gives

$$f'(x) = \frac{1-e^{-x(T-s)} - x(T-s)e^{-x(T-s)}}{\left[1-e^{-x(T-s)}\right]^2}.$$

We let $g(x) = 1 - e^{-x(T-s)} - x(T-s)e^{-x(T-s)}$, then

$$g'(x) = x(T-s)^2 e^{-x(T-s)} > 0.$$

Thus, $g(x)$ is an increasing function and $g(x) > g(0) = 0$. In this case, $f(x)$ is also an increasing function. Then the Lemma is proved. □

We may rewrite $\mathrm{E}[M_s]$ as

$$\begin{aligned}
\mathrm{E}[M_s] &= P(0)\frac{1-e^{-c_0(T-s)}}{1-e^{-c_0T}}\int_s^T \mathrm{E}\left[\left(\frac{c_0}{1-e^{-c_0(T-s)}} - \frac{c_s}{1-e^{-c_s(T-s)}}\right)e^{-\int_0^t r_v dv}\right]dt \\
&= P(0)\frac{1-e^{-c_0(T-s)}}{1-e^{-c_0T}}\int_s^T \mathrm{E}\left[\left(\frac{c_0}{1-e^{-c_0(T-s)}} - \frac{c_s}{1-e^{-c_s(T-s)}}\right)e^{-\int_0^s r_v dv}B(s,t)\right]dt,
\end{aligned}$$

and to simplify the calculation, we assume $c_0 = r_0$.

Theorem 3.2.2. *If c_s is defined by the equation*

$$\frac{c_s}{1-e^{-c_s(T-s)}} = \frac{r_0}{1-e^{-r_0(T-s)}} + \frac{1-e^{-r_0(T-s)} - r_0(T-s)}{\left(1-e^{-r_0(T-s)}\right)r_0(T-s)}(r_0 - r_s), \tag{3.10}$$

thus, (3.9) *can be evaluated as*

$$\mathrm{E}[M_s] = P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \left[r_0 - \frac{\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0\frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}}\right]B(0,t)dt. \tag{3.11}$$

Lemma 3.2.3. *If c_s is defined as in* (3.10), *we have*

$$c_s > r_s. \tag{3.12}$$

Proof. We let $x = r_s(T-s)$, $\quad x_0 = r_0(T-s)$, and $y = c_s(T-s)$. Thus, we have

$$g(x) = \frac{x_0}{1-e^{-x_0}} - \frac{x}{1-e^{-x}}.$$

With $g(x_0) = 0$ and $g(0) = \frac{x_0}{1-e^{-x_0}} - 1$, the slope m is

$$m = \frac{g(0)-g(x_0)}{0-x_0} = \frac{1-e^{-x_0}-x_0}{\left(1-e^{-x_0}\right)x_0}.$$

As $g'(x) < 0$ and $g''(x) < 0$, there exists an unique $y \in (x, x_0]$, such that

$$g(y) = m(x - x_0),$$

thus, we have $c_s > r_s$. □

Theorem 3.2.4. *The optimal time to refinance, with $\rho = 1$, can be obtained by the following equation*

$$
\begin{aligned}
&\frac{(r_0(T-s)+1)\, e^{-r_0(T-s)}-1}{(T-s)\left[e^{-r_0(T-s)}+r_0(T-s)-1\right]} \int_s^T \left[r_0 - \frac{\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0 \frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}} \right] B(0,t)dt \\
=&\left\{ \int_s^T \frac{\frac{\partial^2 \ln(A_1(s,t))}{\partial s \partial t}\frac{dA_2(s,t)}{dt} - \frac{\partial^2 A_2(s,t)}{\partial s \partial t}\left[\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0\frac{dA_2(0,t)}{dt}\right]}{\left[\frac{dA_2(s,t)}{dt}\right]^2} B(0,t)dt \right. \\
&+ \left. \left[r_0 - \frac{\frac{d\ln(A_1(s,t))}{dt}|_{t=s} - \frac{d\ln(A_1(0,t))}{dt}|_{t=s} + r_0\frac{dA_2(0,t)}{dt}|_{t=s}}{\frac{dA_2(s,t)}{dt}|_{t=s}} \right] B(0,s) \right\}. \qquad (3.13)
\end{aligned}
$$

In addition, we can obtain s by numerical methods.

Theorem 3.2.5. *The analytical solution of* $\mathrm{E}[M_s]$ *is obtained when c_t is a linear function of r_t, say, $c_t = \lambda r_t$, where λ is a multiplier, with $\lambda > 1$.*

$$
\mathrm{E}[M_s] = \frac{P(0)}{1-e^{-\lambda r_0 T}} \int_s^T \lambda r_0 B(0,t) - \left[1 - e^{-\lambda r_0 (T-s)}\right] \sum_{n=0}^{\infty} B_n \frac{(-1)^n \lambda^n (T-s)^{n-1} G_\alpha^{(n)}(0,s,t)}{n!} dt,
$$

with $G(\alpha,s,t) = \frac{A_1(s,t)}{A_1(s,\tilde{t})} B(0,\tilde{t})$, and $\tilde{t}$ is a function of α and t. In addition, if $\alpha = 0$, we have $G(0,s,t) = B(0,t)$.

Lemma 3.2.6. *With $c_t = \lambda r_t$, M_s could be rewritten as*

$$
M_s = \frac{P(0)}{1-e^{-\lambda r_0 T}} \int_s^T \lambda r_0 e^{-\int_0^t r_v dv} - \left[1 - e^{-\lambda r_0 (T-s)}\right] \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^{n-1}}{n!} \lambda^n r_s^n e^{-\int_0^t r_v dv} dt, \qquad (3.14)
$$

where B_n is the sequence of Bernoulli numbers with the explicit formula

$$
B_n = \sum_{k=0}^{n} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{(v+1)^n}{k+1}.
$$

Proof. We may arrange M_s as

$$
\begin{aligned}
M_s =& P(0) \left[\frac{c_0}{1-e^{-c_0 T}} - \frac{c_s\left[1-e^{-c_0(T-s)}\right]}{\left[1-e^{-c_0 T}\right]\left[1-e^{-c_s(T-s)}\right]} \right] \int_s^T e^{-\int_0^t r_v dv} dt \\
=& \frac{P(0)}{1-e^{-c_0 T}} \int_s^T c_0 e^{-\int_0^t r_v dv} - \frac{1-e^{-c_0(T-s)}}{T-s} \frac{c_s(T-s)}{1-e^{-c_s(T-s)}} e^{-\int_0^t r_v dv} dt,
\end{aligned}
$$

As we have

$$
\frac{c_s(T-s)}{1-e^{-c_s(T-s)}} = \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^n c_s^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^n \lambda^n r_s^n}{n!},
$$

thus, we may rewrite M_s as

$$M_s = \frac{P(0)}{1-e^{-\lambda r_0 T}} \int_s^T \lambda r_0 e^{-\int_0^t r_v dv} - \left[1-e^{-\lambda r_0 (T-s)}\right] \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^{n-1}}{n!} \lambda^n r_s^n e^{-\int_0^t r_v dv} dt.$$

□

Lemma 3.2.7. *Assume $c_t = \lambda r_t$. Then the asymptotic formula of* (3.9) *can be simplified as*

$$M_s \approx \lambda P(0) \frac{1-e^{-\lambda r_0 (T-s)} - \lambda r_0 (T-s) e^{-\lambda r_0 (T-s)}}{\left[1-e^{-\lambda r_0 (T-s)}\right]\left[1-e^{-\lambda r_0 T}\right]} \int_s^T (r_0 - r_s) e^{-\int_0^t r_v dv} dt, \quad (3.15)$$

thus, the expectation of (3.15) is

$$\begin{aligned} \mathrm{E}[M_s] \approx & \lambda P(0) \frac{1-e^{-\lambda r_0 (T-s)} - \lambda r_0 (T-s) e^{-\lambda r_0 (T-s)}}{\left[1-e^{-\lambda r_0 (T-s)}\right]\left[1-e^{-\lambda r_0 T}\right]} \\ & \int_s^T \left[r_0 - \frac{\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0 \frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}} \right] B(0,t) dt. \end{aligned} \quad (3.16)$$

Lemma 3.2.8. *The approximation of $\frac{c_0}{1-e^{-c_0(T-s)}} - \frac{c_s}{1-e^{-c_s(T-s)}}$ is*

$$\begin{aligned} \frac{c_0}{1-e^{-c_0(T-s)}} - \frac{c_s}{1-e^{-c_s(T-s)}} \approx & \frac{1-e^{-c_0(T-s)} - c_0(T-s)e^{-c_0(T-s)}}{\left[1-e^{-c_0(T-s)}\right]^2}(c_0 - c_s) \\ = & \lambda \frac{1-e^{-\lambda r_0(T-s)} - \lambda r_0(T-s)e^{-\lambda r_0(T-s)}}{\left[1-e^{-\lambda r_0(T-s)}\right]^2}(r_0 - r_s). \end{aligned}$$

However, the approximated value of M_s is slightly higher than the real value, suggesting that our approximation will benefit the debtors. In this approximation method, the lender may charge for a higher transaction cost (with the higher value of β) to keep balance.

Proof. We let $x = c_s(T-s), \quad x_0 = c_0(T-s)$. As $\exists\, A$, such that $0 \le x \le A$, we approximate $f(x) = \frac{x}{1-e^{-x}} - \frac{x_0}{1-e^{-x_0}}$ at the point x_0 as

$$\frac{x}{1-e^{-x}} - \frac{x_0}{1-e^{-x_0}} = \frac{1-e^{-x_0} - x_0 e^{-x_0}}{[1-e^{-x_0}]^2}(x - x_0) + o(x - x_0).$$

As

$$f''(x) = \frac{-2e^{-x} + 2e^{-2x} + xe^{-x} + xe^{-2x}}{[1-e^{-x}]^3} > 0,$$

it is clear that $f(x)$ is a convex function, implying $o(c_s - c_0) > 0$. □

Lemma 3.2.9. *For any given function $g(r_s)$, we have*

$$\mathrm{E}\left[g(r_s) r_s e^{-\int_0^t r_v dv}\right] = -\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}} \frac{d}{dt}\left(\frac{1}{A_1(s,t)} \mathrm{E}\left[g(r_s) e^{-\int_0^t r_v dv}\right]\right).$$

Proof.

$$\begin{aligned}
\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}\Big] =&\mathrm{E}\Big[\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}|_{r_s}\Big]\Big]\\
=&\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}\mathrm{E}\Big[e^{-\int_s^t r_v dv}|_{r_s}\Big]\Big]\\
=&\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}B(s,t)\Big]\\
=&\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}A_1(s,t)e^{-A_2(s,t)r_s}\Big].
\end{aligned}$$

Taking derivative of both sides with respect to t gives

$$\begin{aligned}
&\frac{d}{dt}\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}\Big]\\
=&\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}\frac{dA_1(s,t)}{dt}e^{-A_2(s,t)r_s}\Big]-\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}\frac{dA_2(s,t)}{dt}r_sA_1(s,t)e^{-A_2(s,t)r_s}\Big]\\
=&\frac{d\ln\left[A_1(s,t)\right]}{dt}\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}B(s,t)\Big]-\frac{dA_2(s,t)}{dt}\mathrm{E}\Big[g(r_s)r_se^{-\int_0^s r_v dv}B(s,t)\Big]\\
=&\frac{d\ln\left[A_1(s,t)\right]}{dt}\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}\Big]-\frac{dA_2(s,t)}{dt}\mathrm{E}\Big[g(r_s)r_se^{-\int_0^t r_v dv}\Big].
\end{aligned}$$

Rearranging the above equation gives

$$\begin{aligned}
&\mathrm{E}\Big[g(r_s)r_se^{-\int_0^t r_v dv}\Big]\\
=&\frac{1}{\frac{dA_2(s,t)}{dt}}\left[-\frac{d}{dt}\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}\Big]+\frac{d\ln\left[A_1(s,t)\right]}{dt}\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}\Big]\right]\\
=&-\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}}\frac{d}{dt}\left[\frac{1}{A_1(s,t)}\mathrm{E}\Big[g(r_s)e^{-\int_0^t r_v dv}\Big]\right].
\end{aligned}$$

This proves the Lemma. $\square$

Lemma 3.2.10. *For any α in the positive neighborhood $(0,\epsilon)$, if we let $G(\alpha,s,t)=\mathrm{E}\Big[e^{\alpha r_s}e^{-\int_0^t r_v dv}\Big]$, there exists $\tilde{t}=\tilde{t}(\alpha,t)\in[s,t)$, such that*

$$G(\alpha,s,t)=\frac{A_1(s,t)}{A_1(s,\tilde{t})}B(0,\tilde{t}). \tag{3.17}$$

Then we have

$$\mathrm{E}\Big[r_s^ne^{-\int_0^t r_v dv}\Big]=G_\alpha^{(n)}(0,s,t)=\frac{d^n}{d\alpha^n}|_{\alpha=0}\left(\frac{A_1(s,t)}{A_1(s,\tilde{t})}B(0,\tilde{t})\right). \tag{3.18}$$

Proof. We have

$$\begin{aligned}
G(\alpha,s,t)=&\mathrm{E}\Big[g(r_s)e^{-\int_0^s r_v dv}B(s,t)\Big]\\
=&\mathrm{E}\Big[e^{\alpha r_s}e^{-\int_0^s r_v dv}A_1(s,t)e^{-A_2(s,t)r_s}\Big]\\
=&\mathrm{E}\Big[e^{-\int_0^s r_v dv}A_1(s,t)e^{-(A_2(s,t)-\alpha)r_s}\Big],
\end{aligned}$$

as $A_2(s,t)$ is an increasing function with respect to t, there exists $\tilde{t}$, such that

$$0 \leq A_2(s,t) - \alpha = A_2(s,\tilde{t}), \tag{3.19}$$

and we can solve $\tilde{t}$ based on the function of $A_2(s,t)$. Thus, the above equation can be continued as

$$\begin{aligned}
&= \mathrm{E}\Big[e^{-\int_0^s r_v dv} A_1(s,t) e^{-A_2(s,\tilde{t}) r_s}\Big] \\
&= \frac{A_1(s,t)}{A_1(s,\tilde{t})} \mathrm{E}\Big[e^{-\int_0^s r_v dv} A_1(s,\tilde{t}) e^{-A_2(s,\tilde{t}) r_s}\Big] \\
&= \frac{A_1(s,t)}{A_1(s,\tilde{t})} \mathrm{E}\Big[e^{-\int_0^s r_v dv} B(s,\tilde{t})\Big] \\
&= \frac{A_1(s,t)}{A_1(s,\tilde{t})} B(0,\tilde{t}).
\end{aligned}$$

□

Both Lemma 3.2.9 and Lemma 3.2.10 can be used to calculate $\mathrm{E}\Big[r_s^n e^{-\int_0^t r_v dv}\Big]$ based on affine Models. However, Lemma 3.2.10 is more applicable to analytical analysis, and Lemma 3.2.9 is more suitable for numerical iteration.

Corollary 3.2.11. *The same formulation of* $\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big]$ *can be obtained both by* Lemma 3.2.9 *and* Lemma 3.2.10.

Proof. We first consider $g(r_s) = 1$ in Lemma 3.2.9, where

$$\begin{aligned}
\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big] &= -\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}} \frac{d}{dt}\left(\frac{1}{A_1(s,t)} \mathrm{E}\Big[e^{-\int_0^t r_v dv}\Big]\right) \\
&= -\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}} \frac{d}{dt}\left(\frac{1}{A_1(s,t)} B(0,t)\right) \\
&= \frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}} \frac{\frac{dA_1(s,t)}{dt} B(0,t) - \frac{dB(0,t)}{dt} A_1(s,t)}{A_1^2(s,t)} \\
&= \frac{1}{\frac{dA_2(s,t)}{dt}} \frac{\frac{dA_1(s,t)}{dt} B(0,t) - \frac{dB(0,t)}{dt} A_1(s,t)}{A_1(s,t)}.
\end{aligned}$$

As we have

$$\begin{aligned}
\frac{dB(0,t)}{dt} &= \frac{d}{dt} A_1(0,t) e^{-A_2(0,t) r_0} \\
&= \frac{dA_1(0,t)}{dt} e^{-A_2(0,t) r_0} - r_0 \frac{dA_2(0,t)}{dt} A_1(0,t) e^{-A_2(0,t) r_0} \\
&= \frac{dA_1(0,t)}{dt} \frac{1}{A_1(0,t)} A_1(0,t) e^{-A_2(0,t) r_0} - r_0 \frac{dA_2(0,t)}{dt} A_1(0,t) e^{-A_2(0,t) r_0} \\
&= \frac{d \ln\left(A_1(0,t)\right)}{dt} B(0,t) - r_0 \frac{dA_2(0,t)}{dt} B(0,t),
\end{aligned}$$

Thus,

$$\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big] = \frac{\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0\frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}} B(0,t).$$

Similarly, we apply $n=1$, in Lemma 3.2.10, which gives

$$\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big] = G_\alpha^{(1)}(0,s,t) = \frac{d}{d\alpha}|_{\alpha=0}\left(\frac{A_1(s,t)}{A_1(s,\tilde{t})}B(0,\tilde{t})\right).$$

As one can see that $\tilde{t}$ is a function of α, we may calculate $G_\alpha^{(1)}(0,s,t)$ as

$$G_\alpha^{(1)}(0,s,t) = \frac{dG(\alpha,s,t)}{d\tilde{t}}\frac{d\tilde{t}}{d\alpha}|_{\alpha=0}.$$

Taking derivative with respect to α of both sides for $A_2(s,t)-\alpha = A_2(s,\tilde{t})$ implies

$$-1 = \frac{dA_2(s,\tilde{t})}{d\tilde{t}}\frac{d\tilde{t}}{d\alpha}.$$

As $\alpha = 0$ is equivalent to $\tilde{t} = t$, we have

$$\frac{d\tilde{t}}{d\alpha}|_{\alpha=0} = \frac{d\tilde{t}}{d\alpha}|_{\tilde{t}=t} = -\frac{1}{\frac{dA_2(s,t)}{dt}}.$$

Thus,

$$\begin{aligned}
&G_\alpha^{(1)}(0,s,t)\\
=&-\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}}\frac{d}{d\tilde{t}}|_{\tilde{t}=t}\left(\frac{B(0,\tilde{t})}{A_1(s,\tilde{t})}\right)\\
=&-\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}}\frac{d}{d\tilde{t}}|_{\tilde{t}=t}\left(\frac{A_1(0,\tilde{t})}{A_1(s,\tilde{t})}e^{-A_2(0,\tilde{t})r_0}\right)\\
=&-\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}}\left[\frac{\frac{dA_1(0,\tilde{t})}{d\tilde{t}}A_1(s,\tilde{t}) - \frac{A_1(s,\tilde{t})}{d\tilde{t}}A_1(0,\tilde{t})}{A_1^2(s,\tilde{t})}e^{-A_2(0,\tilde{t})r_0} - r_0\frac{dA_2(s,\tilde{t})}{d\tilde{t}}\frac{A_1(0,\tilde{t})}{A_1(s,\tilde{t})}e^{-A_2(0,\tilde{t})r_0}\right]|_{\tilde{t}=t}\\
=&-\frac{A_1(s,t)}{\frac{dA_2(s,t)}{dt}}\left[\frac{\frac{dA_1(0,t)}{dt}A_1(s,t) - \frac{A_1(s,t)}{dt}A_1(0,t)}{A_1^2(s,t)}e^{-A_2(0,t)r_0} - r_0\frac{dA_2(s,t)}{dt}\frac{A_1(0,t)}{A_1(s,t)}e^{-A_2(0,t)r_0}\right]\\
=&-\frac{1}{\frac{dA_2(s,t)}{dt}}\left[\frac{\frac{dA_1(0,t)}{dt}A_1(s,t) - \frac{A_1(s,t)}{dt}A_1(0,t)}{A_1(s,t)}e^{-A_2(0,t)r_0} - r_0\frac{dA_2(s,t)}{dt}A_1(0,t)e^{-A_2(0,t)r_0}\right]\\
=&\frac{\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0\frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}}B(0,t).
\end{aligned}$$

One can see that we obtain the same formula of $\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big]$ based on Lemma 3.2.9 and Lemma 3.2.10. □

Theorem 3.2.12. (3.5) *can be evaluated as*

$$\begin{aligned}
\mathrm{V}ar[M_s] =& 2\left(P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2 \int_s^T\int_s^t r_0^2\tilde{G}(0,s,\tilde{t})\\
&-2r_0\tilde{G}_\alpha^{(1)}(0,s,\tilde{t}) + \tilde{G}_\alpha^{(2)}(0,s,\tilde{t})dhdt - (\mathrm{E}[M_s])^2,
\end{aligned} \tag{3.20}$$

with

$$
\begin{aligned}
\tilde{G}(0,s,\tilde{t}) =& \frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\hat{B}(0,\tilde{t}) \\
\tilde{G}_\alpha^{(1)}(0,s,\tilde{t}) =& \frac{1}{2}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}) \\
\tilde{G}_\alpha^{(2)}(0,s,\tilde{t}) =& \frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}} - \frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}}\right. \\
&+\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2 - \left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}\right. \\
&\left.\left.-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 - \frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right]\hat{B}(0,\tilde{t}) + \frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0\right. \\
&\left.+\frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}),
\end{aligned}
$$

where $\hat{r}_t = 2r_t$, and $\hat{B}(0,\tilde{t})$ is the bond price under $\hat{r}_t$. $\tilde{t}$ satisfies $\frac{A_2(h,t)}{2} = \hat{A}_2(h,\tilde{t})$.

Lemma 3.2.13. *The formula of M_s^2 is*

$$
M_s^2 = 2\left(P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t\left(r_0^2 - 2r_0r_s + r_s^2\right)e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}dhdt. \tag{3.21}
$$

Proof. As we have

$$
\begin{aligned}
M_s^2 =& \left[m_0P(0) - m_sP(s)\right]^2\left(\int_s^T e^{-\int_0^t r_u du}dt\right)^2 \\
=& \left[m_0P(0) - m_sP(s)\right]^2\int_s^T\int_s^T e^{-\int_0^t r_u du}e^{-\int_0^h r_u du}dhdt \\
=& \left[m_0P(0) - m_sP(s)\right]^2\left(\int_s^T\int_s^t + \int_s^T\int_t^T\right)e^{-\int_0^t r_u du}e^{-\int_0^h r_u du}dhdt \\
=& 2\int_s^T\int_s^t\left[m_0P(0) - m_sP(s)\right]^2 e^{-\int_0^t r_u du}e^{-\int_0^h r_u du}dhdt \\
=& 2\left(P(0)\frac{1-e^{-r_0(T-s)}}{1-e^{-r_0T}}\right)^2\int_s^T\int_s^t\left(\frac{r_0}{1-e^{-r_0(T-s)}} - \frac{c_s}{1-e^{-c_s(T-s)}}\right)^2 e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}dhdt \\
=& 2\left(P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t (r_0 - r_s)^2 e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}dhdt,
\end{aligned}
$$

then the Lemma is proved. □

Theorem 3.2.14. *The analytical solution of* $\mathrm{V}ar[M_s]$ *is obtained when* $c_t = \lambda r_t$*, where*

$$
\begin{aligned}
\mathrm{V}ar[M_s] =& 2\left(\frac{P(0)}{1-e^{-\lambda r_0 T}}\right)^2 \int_s^T \int_s^t \lambda r_0 \tilde{G}(0,s,\tilde{t}) \\
& -2\lambda r_0 \left[1-e^{-\lambda r_0 (T-s)}\right] \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^{n-1}\lambda^n}{n!} \tilde{G}_{\alpha}^{(n)}(0,s,\tilde{t}) \\
& + \left[1-e^{-\lambda r_0 (T-s)}\right]^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n} B_m B_{n-m} \frac{(-1)^n (T-s)^{n-2}\lambda^n}{m!(n-m)!} \tilde{G}_{\alpha}^{(n)}(0,s,\tilde{t}) - (\mathrm{E}[M_s])^2,
\end{aligned}
\tag{3.22}
$$

with $\tilde{G}(\alpha,s,\tilde{t}) = \frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)$*, and* $\tilde{t}_1$ *is a function of* α *and* $\tilde{t}$*. If* $\alpha = 0$*, we have* $\tilde{G}(0,s,\tilde{t}) = \frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\hat{B}(0,\tilde{t})$*.*

Lemma 3.2.15. M_s^2 *can be rewritten as*

$$
\begin{aligned}
M_s^2 =& 2\left(\frac{P(0)}{1-e^{-\lambda r_0 T}}\right)^2 \int_s^T \int_s^t \left(\lambda^2 r_0^2 - 2\lambda r_0 \frac{1-e^{-\lambda r_0 (T-s)}}{T-s} \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^n r_s^n}{n!}\right. \\
& \left. + \left(\frac{1-e^{-\lambda r_0 (T-s)}}{T-s}\right)^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n} B_m B_{n-m} \frac{(-1)^n (T-s)^n \lambda^n r_s^n}{m!(n-m)!}\right) e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv} dh dt.
\end{aligned}
$$

Proof. We may rewrite M_s^2 as

$$
M_s^2 = 2\left(\frac{P(0)}{1-e^{-\lambda r_0 T}}\right)^2 \int_s^T \int_s^t \left(\lambda r_0 - \left[1-e^{-\lambda r_0 (T-s)}\right] \frac{\lambda r_s}{1-e^{-\lambda r_s (T-s)}}\right)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv} dh dt,
$$

and we have

$$
\begin{aligned}
& \left(\lambda r_0 - \left[1-e^{-\lambda r_0 (T-s)}\right] \frac{\lambda r_s}{1-e^{-\lambda r_s (T-s)}}\right)^2 \\
=& \lambda^2 r_0^2 - 2\lambda r_0 \frac{1-e^{-\lambda r_0 (T-s)}}{T-s} \frac{\lambda r_s (T-s)}{1-e^{-\lambda r_s (T-s)}} + \left(\frac{1-e^{-\lambda r_0 (T-s)}}{T-s}\right)^2 \left(\frac{\lambda r_s (T-s)}{1-e^{-\lambda r_s (T-s)}}\right)^2 \\
=& \lambda^2 r_0^2 - 2\lambda r_0 \frac{1-e^{-\lambda r_0 (T-s)}}{T-s} \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^n r_s^n \lambda^n}{n!} \\
& + \left(\frac{1-e^{-\lambda r_0 (T-s)}}{T-s}\right)^2 \left(\sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^n r_s^n \lambda^n}{n!}\right)^2.
\end{aligned}
$$

We apply the Cauchy product to obtain $\left[\sum_{i=0}^{\infty} B_i \frac{(-1)^i (T-s)^i \lambda^i r_s^i}{i!}\right]^2$, which gives

$$
\begin{aligned}
&\left[\sum_{i=0}^{\infty} B_i \frac{(-1)^i (T-s)^i \lambda^i r_s^i}{i!}\right]^2 \\
=&\sum_{i=0}^{\infty} B_i \frac{(-1)^i (T-s)^i \lambda^i r_s^i}{i!} \sum_{j=0}^{\infty} B_j \frac{(-1)^j (T-s)^j \lambda^j r_s^j}{j!} \\
=&\sum_{n=0}^{\infty} \sum_{m=0}^{n} B_m \frac{(-1)^m (T-s)^m \lambda^m r_s^m}{m!} B_{n-m} \frac{(-1)^{n-m} (T-s)^{n-m} \lambda^{n-m} r_s^{n-m}}{(n-m)!} \\
=&\sum_{n=0}^{\infty} \sum_{m=0}^{n} B_m B_{n-m} \frac{(-1)^n (T-s)^n \lambda^n r_s^n}{m!(n-m)!},
\end{aligned}
$$

thus

$$
\begin{aligned}
M_s^2 =&2 \left(\frac{P(0)}{1-e^{-\lambda r_0 T}}\right)^2 \int_s^T \int_s^t \left(\lambda^2 r_0^2 - 2\lambda r_0 \frac{1-e^{-\lambda r_0 (T-s)}}{T-s} \sum_{n=0}^{\infty} B_n \frac{(-1)^n (T-s)^n r_s^n}{n!}\right. \\
&+ \left.\left(\frac{1-e^{-\lambda r_0 (T-s)}}{T-s}\right)^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} B_m B_{n-m} \frac{(-1)^n (T-s)^n \lambda^n r_s^n}{m!(n-m)!}\right) e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv} dh dt.
\end{aligned}
$$

□

Lemma 3.2.16. $\mathrm{E}\Big[e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] = \frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})} \hat{B}(0,\tilde{t})$.

Proof.

$$
\begin{aligned}
\mathrm{E}\Big[e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] &= \mathrm{E}\Big[e^{-2\int_0^h r_v dv} e^{-\int_h^t r_v dv}\Big] \\
&= \mathrm{E}\Big[e^{-2\int_0^h r_v dv} \mathrm{E}\Big[e^{-\int_h^t r_v dv}\Big] |_{r_h}\Big] \\
&= \mathrm{E}\Big[e^{-2\int_0^h r_v dv} B(h,t)\Big] \\
&= A_1(h,t) \mathrm{E}\Big[e^{-2\int_0^h r_v dv} e^{-A_2(h,t) r_h}\Big] \\
&= A_1(h,t) \mathrm{E}\Big[e^{-\int_0^h 2 r_v dv} e^{-\frac{A_2(h,t)}{2} 2 r_h}\Big], \qquad (3.23)
\end{aligned}
$$

by letting $\hat{r}_v = 2r_v$, we can see that $\hat{r}_v$ follows the same distribution with r_v. We denote $\hat{B}(h,t) = \hat{A}_1(h,t) e^{-\hat{A}_2(h,t)\hat{r}_h}$ as the bond price under the process of $\hat{r}$, thus, the above equation can be continued as

$$
= A_1(h,t) \mathrm{E}\Big[e^{-\int_0^h \hat{r}_v dv} e^{-\frac{A_2(h,t)}{2} \hat{r}_h}\Big].
$$

As $A_2(h,t)>0$, there exist $\tilde{t}>h$, such that $\frac{A_2(h,t)}{2}=\hat{A}_2(h,\tilde{t})$, thus, we have

$$\begin{aligned}
&= A_1(h,t)\mathrm{E}\Big[e^{-\int_0^h \hat{r}_v dv}e^{-\hat{A}_2(h,\tilde{t})\hat{r}_h}\Big] \\
&= \frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\mathrm{E}\Big[e^{-\int_0^h \hat{r}_v dv}\hat{B}(h,\tilde{t})\Big] \\
&= \frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\hat{B}(0,\tilde{t}).
\end{aligned}$$

□

Lemma 3.2.17. *For any α in the positive neighborhood $(0,\epsilon)$, if we let $\tilde{G}(\alpha,s,t)=\mathrm{E}\Big[e^{\alpha r_s}e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big]$, there exists $\tilde{t}_1=\tilde{t}_1(\alpha,\tilde{t})\in[s,\tilde{t})$, such that*

$$\tilde{G}(\alpha,s,\tilde{t})=\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1).$$

Thus, we can obtain $\mathrm{E}\Big[r_s^n e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big]=\tilde{G}_\alpha^{(n)}(0,s,\tilde{t})$.

Proof.

$$\begin{aligned}
\mathrm{E}\Big[e^{\alpha r_s}e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big] &= \mathrm{E}\Big[e^{\alpha r_s}e^{-2\int_0^s r_v dv}e^{-2\int_s^h r_v dv}e^{-\int_h^t r_v dv}\Big] \\
&= \mathrm{E}\Big[e^{\alpha r_s}e^{-2\int_0^s r_v dv}\mathrm{E}\Big[e^{-2\int_s^h r_v dv}e^{-\int_h^t r_v dv}\Big]|_{r_s}\Big] \\
&= \mathrm{E}\left[e^{\alpha r_s}e^{-2\int_0^s r_v dv}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\hat{B}(s,\tilde{t})\right] \\
&= \frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\mathrm{E}\Big[e^{\frac{\alpha}{2}\hat{r}_s}e^{-\int_0^s \hat{r}_v dv}e^{-\hat{A}_2(s,\tilde{t})\hat{r}_s}\Big] \\
&= \frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1),
\end{aligned} \tag{3.24}$$

where $\hat{A}_2(s,\tilde{t})-\frac{\alpha}{2}=\hat{A}_2(s,\tilde{t}_1)$. □

Corollary 3.2.18. *Assume $n=1$ in* Lemma 3.2.17. *Then we have*

$$\mathrm{E}\Big[r_s e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big]=\frac{1}{2}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}+\hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}),$$

with $\hat{r}_0=2r_0$.

Proof.

$$\begin{aligned}
\mathrm{E}\Big[r_s e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big] &= \frac{d}{d\alpha}|_{\alpha=0}\tilde{G}(\alpha,s,\tilde{t}) \\
&= \frac{d}{d\tilde{t}_1}\left(\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)\right)\frac{d\tilde{t}_1}{d\alpha}|_{\alpha=0}.
\end{aligned} \tag{3.25}$$

Taking derivative of $\hat{A}_2(s,\tilde{t}) - \frac{\alpha}{2} = \hat{A}_2(s,\tilde{t}_1)$ with respect to α gives

$$-\frac{1}{2} = \frac{\hat{A}_2(s,\tilde{t}_1)}{d\tilde{t}_1}\frac{d\tilde{t}_1}{d\alpha}|_{\alpha=0},$$

thus, we have

$$\frac{d\tilde{t}_1}{d\alpha}|_{\alpha=0} = \frac{d\tilde{t}_1}{d\alpha}|_{\tilde{t}_1=\tilde{t}} = -\frac{1}{2}\frac{1}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}},$$

Hence, we can continue (3.25) as

$$\begin{aligned}
&= -\frac{1}{2}\frac{1}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{d}{d\tilde{t}_1}|_{\tilde{t}_1=\tilde{t}}\left(\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)\right)\\
&= -\frac{1}{2}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{d}{d\tilde{t}_1}|_{\tilde{t}_1=\tilde{t}}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_1(s,\tilde{t}_1)\hat{r}_0}\right)\\
&= \frac{1}{2}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}).
\end{aligned}$$

□

Corollary 3.2.19. *Assume $n = 2$ in* Lemma 3.2.17. *Then we have*

$$\begin{aligned}
&\mathrm{E}\left[r_s^2 e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\right] \qquad (3.26)\\
=&\frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}} - \frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}} + \left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2\right.\\
&\left.-\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 - \frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right]\hat{B}(0,\tilde{t})\\
&+\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 + \frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}).
\end{aligned}$$

Proof.

$$\begin{aligned}
\mathrm{E}\left[r_s^2 e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\right] &= \frac{d^2}{d\alpha^2}|_{\alpha=0}\tilde{G}(\alpha,s,\tilde{t})\\
&= \frac{d^2}{d\tilde{t}_1^2}\left(\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)\right)\left(\frac{d\tilde{t}_1}{d\alpha}\right)^2|_{\alpha=0}\\
&+\frac{d}{d\tilde{t}_1}\left(\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)\right)\frac{d^2\tilde{t}_1}{d\alpha^2}|_{\alpha=0}. \qquad (3.27)
\end{aligned}$$

Taking derivative of $\hat{A}_2(s,\tilde{t}) - \frac{\alpha}{2} = \hat{A}_2(s,\tilde{t}_1)$ with respect to α twice gives

$$0 = \frac{d^2\hat{A}_2(s,\tilde{t}_1)}{d\tilde{t}_1^2}\left(\frac{d\tilde{t}_1}{d\alpha}\right)^2|_{\alpha=0} + \frac{d\hat{A}_2(s,\tilde{t}_1)}{d\tilde{t}_1}\frac{d\tilde{t}_1^2}{d^2\alpha}|_{\alpha=0},$$

thus, we have

$$\frac{d\tilde{t}_1^2}{d^2\alpha}|_{\alpha=0} = -\frac{1}{4}\frac{d^2\hat{A}_2(s,\tilde{t}_1)}{d\tilde{t}_1^2}\left(\frac{1}{\frac{d\hat{A}_2(s,\tilde{t}_1)}{d\tilde{t}_1}}\right)^3|_{\tilde{t}_1=\tilde{t}} = -\frac{1}{4}\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}\left(\frac{1}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right)^3.$$

Hence, we can continue (3.27) as

$$=\frac{d^2}{d\tilde{t}_1^2}\left(\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)\right)\left(\frac{1}{2}\frac{1}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right)^2|_{\tilde{t}_1=\tilde{t}}$$

$$-\frac{1}{4}\frac{d}{d\tilde{t}_1}\left(\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\hat{A}_1(s,\tilde{t}_1)}\hat{B}(0,\tilde{t}_1)\right)\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}\left(\frac{1}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right)^3|_{\tilde{t}_1=\tilde{t}}$$

$$=\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\frac{d^2}{d\tilde{t}_1^2}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}$$

$$-\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^3}\frac{d}{d\tilde{t}_1}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}$$

$$=\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d}{d\tilde{t}_1}\left(\frac{\frac{d\hat{A}_1(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(s,\tilde{t}_1)-\frac{\hat{A}_1(s,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1^2(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)\right.$$

$$\left.-\frac{d^2\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1^2}\hat{r}_0\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(s,\tilde{t}_1)\hat{r}_0}-\frac{d\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{r}_0\frac{d}{d\tilde{t}_1}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)\right]|_{\tilde{t}_1=\tilde{t}}$$

$$-\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^3}\frac{d}{d\tilde{t}_1}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}$$

$$=\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\frac{d}{d\tilde{t}_1}\left(\frac{\frac{d\hat{A}_1(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(s,\tilde{t}_1)-\frac{d\hat{A}_1(s,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1^2(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}$$

$$+\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[-\frac{d^2\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1^2}\hat{r}_0\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(s,\tilde{t}_1)\hat{r}_0}\right]$$

$$-\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{r}_0+\frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\frac{d}{d\tilde{t}_1}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}.$$

As we have

$$
\begin{aligned}
&\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\frac{d}{d\tilde{t}_1}\left(\frac{\frac{d\hat{A}_1(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(s,\tilde{t}_1)-\frac{d\hat{A}_1(s,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1^2(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}\\
=&\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left(\frac{\frac{d^2\hat{A}_1(0,\tilde{t}_1)}{d\tilde{t}_1^2}\hat{A}_1(s,\tilde{t}_1)-\frac{d^2\hat{A}_1(s,\tilde{t}_1)}{d^2\tilde{t}_1^2}\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1^2(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}+\left(\frac{d\hat{A}_1(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(s,\tilde{t}_1)\right.\right.\\
&\left.\left.-\frac{d\hat{A}_1(s,\tilde{t}_1)}{d\tilde{t}_1}\hat{A}_1(0,\tilde{t}_1)\right)\frac{-\frac{d\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{r}_0\hat{A}_1^2(s,\tilde{t}_1)-2\hat{A}_1(s,\tilde{t}_1)\frac{d\hat{A}_1(s,\tilde{t}_1)}{d\tilde{t}_1}}{\hat{A}_1^4(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}\\
=&\frac{1}{4}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}}-\frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}}+\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}\right)^2\right.\\
&\left.-\left(\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2+\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\left(-\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0-2\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\right]\\
=&\frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}}-\frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}}+\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2\right.\\
&\left.-\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0\right]\hat{B}(0,\tilde{t}),
\end{aligned}
$$

and

$$
\begin{aligned}
&=\frac{1}{4}\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[-\frac{d^2\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1^2}\hat{r}_0\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(s,\tilde{t}_1)\hat{r}_0}\right]|_{\tilde{t}_1=\tilde{t}}\\
&=\frac{1}{4}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[-\frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right],
\end{aligned}
$$

and based on Corollary 3.2.18, we have

$$
\begin{aligned}
&-\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{r}_0+\frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)\hat{A}_1(s,\tilde{t})}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\frac{d}{d\tilde{t}_1}\left(\frac{\hat{A}_1(0,\tilde{t}_1)}{\hat{A}_1(s,\tilde{t}_1)}e^{-\hat{A}_2(0,\tilde{t}_1)\hat{r}_0}\right)|_{\tilde{t}_1=\tilde{t}}\\
&=\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t}_1)}{d\tilde{t}_1}\hat{r}_0+\frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}+\hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}).
\end{aligned}
$$

Thus, we can obtain

$$
\begin{aligned}
&\mathrm{E}\left[r_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right] \\
=&\frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}} - \frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}} + \left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2\right. \\
&\left.-\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 - \frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right]\hat{B}(0,\tilde{t}) \\
&+\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 + \frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}).
\end{aligned}
$$

□

3.3 The Relationship Between Discrete Case and Contines Case

We adopt the same assumption for the discrete case that m_s represent the monthly payment instead of the continuous payment rate. We consider matching the repayment of principal and interest method, in which a fixed amount of payment is made in each month during the whole period of the mortgage contract. The typical settings in such a scheme is that the principal is to be paid back over a period of $12T$ months. The first payment is made at month 1, and the last at month $12T$. In each month a fixed amount of payment m_0 is made and this monthly payment rate m_0 is calculated by

$$
m_0 = \frac{\frac{c_0}{12}}{1-\left(1+\frac{c_0}{12}\right)^{-12T}}, \tag{3.28}
$$

At the $12s$th month, $s \in \{1, 2, \ldots, T\}$, in this scheme, after the monthly payment m_0 has been made, the outstanding balance $P(s)$ owed to the lender is

$$
P(s) = \frac{12m_0P(0)}{c_0}\left[1-\left(1+\frac{c_0}{12}\right)^{-12(T-s)}\right]. \tag{3.29}
$$

The value of the portfolio V consisting of a loan P(0) and a refinancing agreement is

$$
M_s = [m_0P(0) - m_sP(s)]\sum_{i=s}^{T}\frac{1}{\prod_{j=0}^{i}\left[1+\frac{r_j}{12}\right]}, \tag{3.30}
$$

where

$$
m_0P(0) - m_sP(s) = P(0)\left[\frac{\frac{c_0}{12}}{1-\left(1+\frac{c_0}{12}\right)^{-12T}} - \frac{\frac{c_s}{12}\left[1-\left(1+\frac{c_0}{12}\right)^{-12(T-s)}\right]}{\left[1-\left(1+\frac{c_0}{12}\right)^{-12T}\right]\left[1-\left(1+\frac{c_s}{12}\right)^{-12(T-s)}\right]}\right].
$$

If the payment scheme is continuous, (3.30) can be transformed to

$$
\begin{aligned}
&M_s \\
=& \lim_{N\to\infty} P(0)\left[\frac{\frac{c_0}{N}}{1-\left(1+\frac{c_0}{N}\right)^{-NT}} - \frac{\frac{c_s}{N}\left[1-\left(1+\frac{c_0}{N}\right)^{-N(T-s)}\right]}{\left[1-\left(1+\frac{c_0}{N}\right)^{-NT}\right]\left[1-\left(1+\frac{c_s}{N}\right)^{-N(T-s)}\right]}\right] \\
&\sum_{i=s}^{T}\frac{1}{\prod_{j=0}^{i}\left[1+\frac{r_j}{N}\right]} \\
=& \lim_{N\to\infty} P(0)\left[\frac{c_0}{1-\left(1+\frac{c_0}{N}\right)^{-NT}} - \frac{c_s\left[1-\left(1+\frac{c_0}{N}\right)^{-N(T-s)}\right]}{\left[1-\left(1+\frac{c_0}{N}\right)^{-NT}\right]\left[1-\left(1+\frac{c_s}{N}\right)^{-N(T-s)}\right]}\right] \\
&\sum_{i=s}^{T}\frac{1}{\prod_{j=0}^{i}\left[1+\frac{r_j}{N}\right]}\frac{1}{N} \\
=&P(0)\left[\frac{c_0}{1-e^{-c_0T}} - \frac{c_s\left[1-e^{-c_0(T-s)}\right]}{\left[1-e^{-c_0T}\right]\left[1-e^{-c_s(T-s)}\right]}\right]\lim_{N\to\infty}\sum_{i=s}^{T}\frac{1}{\prod_{j=0}^{i}\left[1+\frac{r_j}{N}\right]}\frac{1}{N}. \qquad (3.31)
\end{aligned}
$$

As

$$
\begin{aligned}
\lim_{N\to\infty}\ln\left[\frac{1}{\prod_{j=0}^{i}\left[1+\frac{r_j}{N}\right]}\right] &= -\lim_{N\to\infty}\sum_{j=0}^{i}\ln\left[1+\frac{r_j}{N}\right] \\
&= -\lim_{N\to\infty}\sum_{j=0}^{i}\frac{r_j}{N} \\
&= -\int_0^i r_u du,
\end{aligned}
$$

(3.31) is continued as

$$
\begin{aligned}
=&P(0)\left[\frac{c_0}{1-e^{-c_0T}} - \frac{c_s\left[1-e^{-c_0(T-s)}\right]}{\left[1-e^{-c_0T}\right]\left[1-e^{-c_s(T-s)}\right]}\right]\lim_{N\to\infty}\sum_{i=s}^{T}e^{-\int_s^i r_u du}\frac{1}{N} \\
=&P(0)\left[\frac{c_0}{1-e^{-c_0T}} - \frac{c_s\left[1-e^{-c_0(T-s)}\right]}{\left[1-e^{-c_0T}\right]\left[1-e^{-c_s(T-s)}\right]}\right]\int_s^T e^{-\int_0^t r_u du}dt \\
=&\left[m_0P(0)-m_sP(s)\right]\int_s^T e^{-\int_0^t r_u du}dt.
\end{aligned}
$$

Chapter 4

Results for Various Models of Interest Rate

4.1 Merton Model

Merton([27]) proposed a general stochastic process to describe the evolution of interest rate dynamic. The explicit solution of r_t is

$$r_t = r_0 + ut + \sigma W_t. \tag{4.1}$$

4.1.1 $T < \infty$

Recall that we have

$$\mathrm{E}[M_s] = P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1 - e^{-r_0 T}\right]} \int_s^\infty \mathrm{E}\Big[(r_0 - r_s)e^{-\int_0^t r_v dv}\Big] dt.$$

As $r_0 - r_s = -us - \sigma W_s$, we may continue the calculation of $\mathrm{E}\Big[(r_0 - r_s)e^{-\int_0^t r_v dv}\Big]$ as

$$\begin{aligned}
\mathrm{E}\Big[(r_0 - r_s)e^{-\int_0^t r_v dv}\Big] &= -us\mathrm{E}\Big[e^{-\int_0^t r_v dv}\Big] - \sigma\mathrm{E}\Big[W_s e^{-\int_0^t r_v dv}\Big] \\
&= -usB(0,t)dt - \sigma\mathrm{E}\Big[W_s e^{-\int_0^t r_v dv}\Big],
\end{aligned}$$

where

$$B(0,t) = e^{-r_0 t - \frac{ut^2}{2} + \frac{\sigma^2 t^3}{6}}.$$

We can rearrange $\mathrm{E}\Big[W_s e^{-\int_0^t r_v dv}\Big]$ as

$$\begin{aligned}
\mathrm{E}\Big[W_s e^{-\int_0^t r_v dv}\Big] =& \mathrm{E}\Big[W_s e^{-\int_0^t r_0 + uv + \sigma W_v dv}\Big] \\
=& e^{-r_0 t - \frac{1}{2}ut^2}\mathrm{E}\Big[W_s e^{-\sigma\int_0^t W_v dv}\Big] \\
=& e^{-r_0 t - \frac{1}{2}ut^2}\mathrm{E}\Big[W_s e^{-\sigma t W_t} e^{\sigma\int_0^t v dW_v}\Big] \\
=& e^{-r_0 t - \frac{1}{2}ut^2}\mathrm{E}\Big[W_s e^{-\sigma(tW_t - tW_s)} e^{-\sigma t W_s} e^{\sigma\int_0^s v dW_v} e^{\sigma\int_s^t v dW_v}\Big] \\
=& e^{-r_0 t - \frac{1}{2}ut^2}\mathrm{E}\Big[W_s e^{-\sigma t W_s} e^{\sigma\int_0^s v dW_v}\Big]\mathrm{E}\Big[e^{-\sigma(tW_t - tW_s)} e^{\sigma\int_s^t v dW_v}\Big].
\end{aligned}$$

Thus, as

$$\mathrm{E}\Big[e^{-\sigma t W_s} e^{\sigma \int_0^s v dW_v}\Big] = \mathrm{E}\Big[e^{-\sigma \int_0^s (t-v) dW_v}\Big],$$

we let $q = \int_0^s (t-v)dW_v$. It is well known that q follows normal distribution. Clearly, we have

$$\mathrm{E}[q] = 0,$$

and by adopting the Ito's isometry, we obtain

$$\begin{aligned}
\mathrm{Var}[q] =& \mathrm{E}[q^2] - [\mathrm{E}[q]]^2 \\
=& \mathrm{E}[q^2] \\
=& \mathrm{E}\left[\left(\int_0^s (t-v)dW_v\right)^2\right] \\
=& \int_0^s (t-v)^2 dv \\
=& \frac{1}{3}t^3 - \frac{1}{3}(t-s)^3,
\end{aligned}$$

which implies $q \sim N(0, \frac{1}{3}t^3 - \frac{1}{3}(t-s)^3)$. Therefore, we have

$$\mathrm{E}\Big[e^{-\sigma \int_s^t (t-v)dW_v}\Big] = M_q(-\sigma) = e^{\frac{\sigma^2\left[t^3-(t-s)^3\right]}{6}}.$$

And we can obtain $\mathrm{E}\Big[W_s e^{-\sigma t W_s} e^{\sigma \int_0^s v dW_v}\Big]$ as

$$\begin{aligned}
\mathrm{E}\Big[W_s e^{-\sigma t W_s} e^{\sigma \int_0^s v dW_v}\Big] =& -\frac{1}{\sigma}\frac{d\mathrm{E}\Big[e^{-\sigma t W_s} e^{\sigma \int_0^s v dW_v}\Big]}{dt} \\
=& -\frac{1}{\sigma}\frac{de^{\frac{\sigma^2\left[t^3-(t-s)^3\right]}{6}}}{dt} \\
=& -\frac{\sigma\left[t^2-(t-s)^2\right]}{2} e^{\frac{\sigma^2\left[t^3-(t-s)^3\right]}{6}}.
\end{aligned}$$

Similarity, we have

$$\begin{aligned}
\mathrm{E}\Big[e^{-\sigma(tW_t - tW_s)} e^{\sigma \int_s^t v dW_v}\Big] =& \mathrm{E}\Big[e^{-\sigma \int_s^t (t-v)dW_v}\Big] \\
=& e^{\frac{\sigma^2 (t-s)^3}{6}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{E}[M_s] =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \mathrm{E}\Big[(r_0-r_s)e^{-\int_0^t r_v dv}\Big]dt \\
=& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \left[-us+\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t)dt \\
=& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \left[r_0-\mathrm{E}[r_s]+\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t)dt.
\end{aligned} \tag{4.2}$$

We apply Corollary 3.2.11 to check if the solution is consistent with our calculation. As in the Merton model, we have with

$$\begin{aligned}
\frac{d\left(\ln A_1(s,t)\right)}{dt} =& -u(t-s)+\frac{1}{2}\sigma^2(t-s)^2 \\
\frac{dA_2(s,t)}{dt} =& 1 \\
\frac{d\left(\ln A_1(0,t)\right)}{dt} =& -ut+\frac{1}{2}\sigma^2t^2 \\
\frac{dA_2(0,t)}{dt} =& 1.
\end{aligned}$$

Thus, we can compute $\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big]$ as

$$\begin{aligned}
\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big] =& \frac{\frac{d\ln(A_1(s,t))}{dt}-\frac{d\ln(A_1(0,t))}{dt}+r_0\frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}}B(0,t) \\
=& \left[-u(t-s)+\frac{1}{2}\sigma^2(t-s)^2+ut-\frac{1}{2}\sigma^2t^2+r_0\right]B(0,t) \\
=& \left[r_0+us-\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathrm{E}[M_s] =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \mathrm{E}\Big[(r_0-r_s)e^{-\int_0^t r_v dv}\Big]dt \\
=& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \left[r_0-\mathrm{E}[r_s]+\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t)dt.
\end{aligned} \tag{4.3}$$

One can see that we obtain the same formula in (4.2) and (4.3).

Figure 4.1 demonstrates the numerical value of $\mathrm{E}[M_s]$ under Merton model. We can see that the values of $\mathrm{E}[M_s]$ change significantly with the variation of T. The results are not surprising in Merton model. The interest rate can be infinite due to the unboundedness of the first and the second moment of r_t. Moreover, when we consider the bond price under Merton model, it is clear that the bond price is an increasing function with the maturity date. In the following section, we will see that the value of $\mathrm{E}[M_s]$ approaches to infinity with the infinite maturity date, which is unrealistic.

Note that

$$\mathrm{E}\big[M_s^2\big] = 2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2 \int_s^\infty \int_s^t \mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] dhdt,$$

thus, we need to compute the value of $\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big]$, where

$$\begin{aligned}
&\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&\mathrm{E}\Big[(us+\sigma W_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&u^2s^2\mathrm{E}\Big[e^{-\int_0^h r_v du} e^{-\int_0^t r_v dv}\Big] + 2us\sigma\mathrm{E}\Big[W_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] + \sigma^2\mathrm{E}\Big[W_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big].
\end{aligned}$$

As we have

$$\begin{aligned}
\mathrm{E}\Big[e^{-\int_0^h r_v du} e^{-\int_0^t r_v du}\Big] =& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2}\mathrm{E}\Big[e^{-\sigma(hW_h+tW_t)} e^{\sigma\int_0^h W_v dv} e^{\sigma\int_0^t W_v dv}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2}\mathrm{E}\Big[e^{-\sigma(hW_h+tW_t-tW_h+tW_h)} e^{2\sigma\int_0^h v dW_v} e^{\sigma\int_h^t v dW_v}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2}\mathrm{E}\Big[e^{-\sigma\int_0^h (h+t-2v) dW_v}\Big]\mathrm{E}\Big[e^{-\sigma\int_h^t (t-v) dW_v}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2} e^{\frac{\sigma^2\left[(t+h)^3-(t-h)^3\right]}{12}} e^{\frac{\sigma^2(t-h)^3}{6}} \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}},
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}\Big[W_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2}\mathrm{E}\Big[W_s e^{-\sigma(hW_h+tW_t)} e^{2\sigma\int_0^h v dW_v} e^{\sigma\int_h^t v dW_v}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2}\mathrm{E}\Big[W_s e^{-\sigma((h+t)(W_h-W_s)+t(W_t-W_h)+(h+t)W_s)} e^{2\sigma\int_0^h v dW_v} e^{\sigma\int_h^t v dW_v}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2}\mathrm{E}\Big[W_s e^{-\sigma\int_0^s (h+t-2v) dW_v}\Big]\mathrm{E}\Big[e^{-\sigma\int_s^h (h+t-2v) dW_v}\Big]\mathrm{E}\Big[e^{-\sigma\int_h^t (t-v) dW_v}\Big] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h-2s)^3+(t-h)^3\right]}{12}}\left[-\frac{1}{\sigma}\frac{d\mathrm{E}\Big[e^{-\sigma\int_0^s (h+t-2v) dW_v}\Big]}{d(h+t)}\right] \\
=& e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h-2s)^3+(t-h)^3\right]}{12}}\left[-\frac{1}{\sigma}\frac{de^{\frac{\sigma^2\left[(t+h)^3-(t+h-2s)^3\right]}{12}}}{d(h+t)}\right] \\
=& -\sigma\frac{(h+t)^2-(h+t-2s)^2}{4} e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h-2s)^3+(t-h)^3\right]}{12}+\frac{\sigma^2\left[(t+h)^3-(t+h-2s)^3\right]}{12}} \\
=& -\sigma s(t+h-s) e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}},
\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{E}\Big[W_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=& e^{-r_0 t-\frac{1}{2}ut^2-r_0 h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h-2s)^3+(t-h)^3\right]}{12}} \left[\frac{1}{\sigma^2}\frac{d^2 e^{\frac{\sigma^2\left[(t+h)^3-(t+h-2s)^3\right]}{12}}}{d(h+t)^2}\right] \\
=& \left[s+\sigma^2 s^2(t+h-s)^2\right] e^{-r_0 t-\frac{1}{2}ut^2-r_0 h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}}.
\end{aligned}
$$

Hence, the value of $\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big]$ is

$$
\begin{aligned}
\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] =& \left[u^2 s^2 - 2u\sigma^2 s^2(t+h-s)+\sigma^2 s+\sigma^4 s^2(t+h-s)^2\right] \\
& e^{-r_0 t-\frac{1}{2}ut^2-r_0 h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}}.
\end{aligned}
$$

Besides, we can apply Lemma 3.2.16, Corollary 3.2.18 and 3.2.19 to Merton model. As the SDE of interest rate in Merton model in terms of $\hat{r}_t$ in Lemma 3.2.16 is

$$d\hat{r}_t = \hat{u}dt + \hat{\sigma}dW_t,$$

thus, we have $\hat{u} = 2u$ and $\hat{\sigma} = 2\sigma$. The bond price under $\hat{r}_t$ is

$$\hat{B}(h,t) = \hat{A}_1(h,t)e^{-\hat{A}_2(h,t)\hat{r}_h},$$

with

$$
\begin{aligned}
\hat{A}_1(h,t) =& \exp\left(-\frac{\hat{u}(t-h)^2}{2}+\frac{\hat{\sigma}^2(t-h)^3}{6}\right) = \exp\left(-u(t-h)^2+\frac{2\sigma^2(t-h)^3}{3}\right) \\
\hat{A}_2(h,t) =& t-h.
\end{aligned}
$$

As $\frac{A_2(h,t)}{2} = \hat{A}_2(h,\tilde{t})$ gives $\tilde{t} = \frac{t+h}{2}$, based on Lemma 3.2.16, we have

$$
\begin{aligned}
&\mathrm{E}\Big[e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=& \frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\hat{B}(0,\tilde{t}) \\
=& \frac{A_1(h,t)\hat{A}_1(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})} e^{-\hat{A}_2(0,\tilde{t})\hat{r}_0} \\
=& \exp\left(-\frac{u(t-h)^2}{2}+\frac{\sigma^2(t-h)^3}{6}-u\tilde{t}^2+\frac{2\sigma^2\tilde{t}^3}{3}+u(\tilde{t}-h)^2-\frac{2\sigma^2(\tilde{t}-h)^3}{3}\right)e^{-\hat{r}_0\tilde{t}} \\
=& \exp\left(-\frac{u(t-h)^2}{2}+\frac{\sigma^2(t-h)^3}{6}-u\frac{(t+h)^2}{4}+\frac{\sigma^2(t+h)^3}{12}+u\frac{(t-h)^2}{4}-\frac{\sigma^2(t-h)^3}{12}\right)e^{-\hat{r}_0\tilde{t}} \\
=& \exp\left(-u\frac{(t+h)^2}{4}+\frac{\sigma^2(t+h)^3}{12}-u\frac{(t-h)^2}{4}+\frac{\sigma^2(t-h)^3}{12}\right)e^{-r_0 h-r_0 t} \\
=& e^{-r_0 t-\frac{1}{2}ut^2-r_0 h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}}.
\end{aligned}
$$

We apply Corollary 3.2.18 to Merton model, and this yields

$$
\begin{aligned}
&\mathrm{E}\Big[r_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&\frac{1}{2}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}) \\
=&\frac{1}{2}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left(-\hat{u}(\tilde{t}-s) + \frac{1}{2}\hat{\sigma}^2(\hat{t}-s)^2 + \hat{u}\tilde{t} - \frac{1}{2}\hat{\sigma}^2\hat{t} + \hat{r}_0\right) \\
=&\frac{1}{2}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left(2us - 2\tilde{t}s\sigma^2 + s^2\sigma^2 + 2r_0\right) \\
=&\left(us - s\sigma^2(t+h-s) + r_0\right) e^{-r_0 t - \frac{1}{2}ut^2 - r_0 h - \frac{1}{2}uh^2 + \frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}},
\end{aligned}
$$

and Corollary 3.2.19 gives

$$
\begin{aligned}
&\mathrm{E}\Big[r_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&\frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}} - \frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}} + \left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2\right. \\
&\left. - \left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 - \frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right]\hat{B}(0,\tilde{t}) \\
&+\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0 + \frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}} - \frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}} + \hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}) \\
=&\frac{1}{4}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\Big[-2u\tilde{t} + 4\sigma^2\tilde{t} + 2u\tilde{t} - 4\sigma^2\left(\tilde{t}-s\right)^2 + 4\left(us - s\sigma^2(t+h-s)\right)^2 \\
&+\left(2us - 2s\sigma^2(t+h-s)\right)\hat{r}_0\Big] + \frac{1}{4}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\Big[\hat{r}_0\left(\hat{r}_0 + 2us - 2s\sigma^2(t+h-s)\right)\Big] \\
=&\Big[\sigma^2 s + u^2s^2 + 2r_0us - 2u\sigma^2s^2(h+t-s) - 2r_0\sigma^2s(h+t-s) + \sigma^4s^2(h+t-s)^2 + r_0^2\Big] \\
&e^{-r_0 t - \frac{1}{2}ut^2 - r_0 h - \frac{1}{2}uh^2 + \frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\mathrm{E}\Big[(r_0 - r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&r_0^2\mathrm{E}\Big[e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] - 2r_0\mathrm{E}\Big[r_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] + \mathrm{E}\Big[r_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&\left[u^2s^2 - 2us^2\sigma^2(t+h-s) + \sigma^2 s + \sigma^4s^2(h+t-s)^2\right] e^{-r_0 t - \frac{1}{2}ut^2 - r_0 h - \frac{1}{2}uh^2 + \frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}}.
\end{aligned}
$$

Thus, $\mathrm{V}ar[M_s]$ can be given by

$$
\begin{aligned}
&\mathrm{V}ar[M_s]\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t \mathrm{E}\Big[(r_0-r_s)^2e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big]dhdt-(\mathrm{E}[M_s])^2\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t \big[u^2s^2-2us^2\sigma^2(t+h-s)+\sigma^2 s\\
&+\sigma^4s^2(h+t-s)^2\big]\,e^{-r_0t-\frac{1}{2}ut^2-r_0h-\frac{1}{2}uh^2+\frac{\sigma^2\left[(t+h)^3+(t-h)^3\right]}{12}}dhdt-(\mathrm{E}[M_s])^2\,,
\end{aligned}
$$

where

$$
\begin{aligned}
[\mathrm{E}[M_s]]^2=&\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\left(\int_s^T\left[-us+\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t)dt\right)^2\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t\left(\left[-us+\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t)\right)\\
&\left(\left[-us+\sigma^2\frac{\left[h^2-(h-s)^2\right]}{2}\right]B(0,h)\right)dhdt\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t\Big(u^2s^2-us^2\sigma^2(t+h-s)\\
&+\frac{s^2\sigma^4}{4}\left(4th-2ts-2hs+s^2\right)\Big)B(0,t)B(0,h)dhdt.
\end{aligned}
$$

4.1.2 $T=\infty$

The value of $\mathrm{E}[M_s]$ can be evaluated as followings with $T=\infty$.

$$
\begin{aligned}
\mathrm{E}[M_s]=&P(0)\int_s^\infty \mathrm{E}\Big[(r_0-r_s)e^{-\int_0^t r_v dv}\Big]dt\\
=&P(0)\int_s^\infty\left[r_0-\mathrm{E}[r_s]+\sigma^2\frac{\left[t^2-(t-s)^2\right]}{2}\right]B(0,t)dt. \qquad (4.4)
\end{aligned}
$$

We can see that the integration in (4.4) will be nonconvergent as $T=\infty$. And the variance of M_s is

$$
\begin{aligned}
&\mathrm{V}ar[M_s]\\
=&2P^2(0)\int_s^\infty\int_s^t \mathrm{E}\Big[(r_0-r_s)^2e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big]-\Big(u^2s^2-us^2\sigma^2(t+h-s)\\
&+\frac{s^2\sigma^4}{4}\left(4th-2ts-2hs+s^2\right)\Big)B(0,t)B(0,h)dhdt\\
=&2P^2(0)\int_s^\infty\int_s^t\Big[\big[u^2s^2-2us^2\sigma^2(t+h-s)+\sigma^2s+\sigma^4s^2(h+t-s)^2\big]\,e^{-\frac{\sigma^2h^3}{6}}\\
&-\left(u^2s^2-us^2\sigma^2(t+h-s)+\frac{s^2\sigma^4}{4}\left(4th-2ts-2hs+s^2\right)\right)\Big]\,B(0,t)B(0,h)dhdt.
\end{aligned}
$$

4.2 Vasicek Model

4.2.1 $T < \infty$

Vasicek ([39]) proposes a model to capture the dynamic of interest rate by a mean-reverting process

$$dr_t = k(\theta - r_t)\,dt + \sigma dW_t, \tag{4.5}$$

where reversion rate k, long-term mean level θ, volatility σ are positive constants, and W_t is the standard Brownian process. Recall that

$$\mathrm{E}[M_s] = P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T \mathrm{E}\Big[(r_0 - r_s)e^{-\int_0^t r_v dv}\Big]dt.$$

Integrating both sides of (4.5) gives

$$r_t - r_0 = k\theta t - k\int_0^t r_v dv + \sigma W_t,$$

or equivalently,

$$\int_0^t r_v dv = \frac{\sigma W_t + k\theta t - r_t + r_0}{k}.$$

Since the explicit solution of (4.5) is given by

$$r_t = \theta + (r_0 - \theta)e^{-kt} + \sigma e^{-kt}\int_0^t e^{ku} dW_u,$$

we may rewrite $\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big]$ as

$$\begin{aligned}
&\mathrm{E}\Big[r_s e^{-\int_0^t r_v dv}\Big]\\
=&e^{-\frac{(r_0-\theta)\left(1-e^{-kt}\right)+k\theta t}{k}}\,\mathrm{E}\left[r_s e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right]\\
=&e^{-\frac{(r_0-\theta)\left(1-e^{-kt}\right)+k\theta t}{k}}\,\mathrm{E}\left[\left[\theta + (r_0-\theta)e^{-ks} + \sigma e^{-ks}\int_0^s e^{kv}dW_v\right]e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right]\\
=&e^{-\frac{(r_0-\theta)\left(1-e^{-ks}\right)+k\theta t}{k}}\left[\left(\theta + (r_0-\theta)e^{-ks}\right)\mathrm{E}\left[e^{\int_0^t f(v)dW_v}\right]\right.\\
&\left.+\mathrm{E}\left[\int_0^s g(v)dW_v e^{\int_0^s f(v)dW_v}\right]\mathrm{E}\left[e^{\int_s^t f(v)dW_v}\right]\right],
\end{aligned}$$

where $f(v) = -\frac{\sigma}{k}\left(1-e^{-k(t-v)}\right)$ and $g(v) = \sigma e^{-k(s-v)}$.

Lemma 4.2.1. *For $n \geq 1$, we have*

$$\mathrm{E}\left[\left(\int_0^s g(v)dW_v\right)^n e^{\int_0^s f(v)dW_v}\right] = \frac{d^n}{d\alpha^n}|_{\alpha=0} e^{\int_0^s \frac{(\alpha g(v)+f(v))^2}{2}dv}.$$

Proof. As we have

$$\begin{aligned}\mathrm{E}\left[e^{\alpha\int_0^s g(v)dW_v}e^{\int_0^s f(v)dW_v}\right] =&\mathrm{E}\left[e^{\int_0^s(\alpha g(v)+f(v))dW_v}\right]\\ =&e^{\int_0^s\frac{(\alpha g(v)+f(v))^2}{2}dv},\end{aligned}$$

thus

$$\begin{aligned}\mathrm{E}\left[\left(\int_0^s g(v)dW_v\right)^n e^{\int_0^s f(v)dW_v}\right] =&\frac{d^n}{d\alpha^n}|_{\alpha=0}\mathrm{E}\left[e^{\alpha\int_0^s g(v)dW_v}e^{\int_0^s f(v)dW_v}\right]\\ =&\frac{d^n}{d\alpha^n}|_{\alpha=0}e^{\int_0^s\frac{(\alpha g(v)+f(v))^2}{2}dv}.\end{aligned}$$

□

Based on Lemma 4.2.1, we have

$$\begin{aligned}&\mathrm{E}\left[\int_0^s g(v)dW_v e^{\int_0^s f(v)dW_v}\right]\\ =&\frac{d}{d\alpha}|_{\alpha=0}e^{\int_0^s\frac{(\alpha g(v)+f(v))^2}{2}dv}\\ =&\int_0^s g(v)\left(\alpha g(v)+f(v)\right)dv e^{\int_0^s\frac{(\alpha g(v)+f(v))^2}{2}dv}|_{\alpha=0}\\ =&\int_0^s g(v)f(v)dv e^{\int_0^s\frac{[f(v)]^2}{2}dv}\\ =&-\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\exp\left[\frac{\sigma^2}{2k^2}\left(s-2\frac{e^{-k(t-s)}-e^{-kt}}{k}+\frac{e^{-2k(t-s)}-e^{-2kt}}{2k}\right)\right].\end{aligned}$$

As one can see, $\int_a^b f(v)dW_v$ follows a normal distribution, with

$$\begin{aligned}\mathrm{E}\left[\int_a^b f(v)dW_v\right] =&0\\ \mathrm{V}ar\left[\int_a^b f(v)dW_v\right] =&\int_a^b f^2(v)dv.\end{aligned}$$

Thus, $\mathrm{E}\left[e^{\int_a^b f(v)dW_v}\right]$ is the moment generating function of $\int_a^b f(v)dW_v$, implying that

$$\mathrm{E}\left[e^{\int_a^b f(v)dW_v}\right] = e^{\frac{\int_a^b f^2(v)dv}{2}},$$

Therefore, we have

$$\begin{aligned}\mathrm{E}\left[e^{\int_0^t f(v)dW_v}\right] =&\exp\left[\frac{\sigma^2}{2k^2}\left(t+\frac{2e^{-kt}}{k}-\frac{e^{-2kt}}{2k}-\frac{3}{2k}\right)\right]\\ \mathrm{E}\left[e^{\int_s^t f(v)dW_v}\right] =&\exp\left[\frac{\sigma^2}{2k^2}\left(t-s+2\frac{e^{-k(t-s)}}{k}-\frac{e^{-2k(t-s)}}{2k}-\frac{3}{2k}\right)\right],\end{aligned}$$

we can obtain $\mathrm{E}\left[r_s e^{-\int_0^t r_v dv}\right]$ as

$$\begin{aligned}
\mathrm{E}\left[r_s e^{-\int_0^t r_v dv}\right] =& e^{-\frac{(r_0-\theta)\left(1-e^{-kt}\right)+k\theta t}{k}}\Bigg[\left(\theta+(r_0-\theta)e^{-ks}\right)e^{\frac{\sigma^2}{2k^2}\left(t+\frac{2e^{-kt}}{k}-\frac{e^{-2kt}}{2k}-\frac{3}{2k}\right)} \\
&-\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)e^{\frac{\sigma^2}{2k^2}\left(t+\frac{2e^{-kt}}{k}-\frac{e^{-2kt}}{2k}-\frac{3}{2k}\right)}\Bigg] \\
=& \left(\theta+(r_0-\theta)e^{-ks}\right)B(0,t)-\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)B(0,t).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathrm{E}[M_s] =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T\Bigg[r_0-\left(\theta+(r_0-\theta)e^{-ks}\right) \\
&+\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\Bigg]B(0,t)dt \\
=& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T\Bigg[r_0-\mathrm{E}[r_s] \\
&+\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\Bigg]B(0,t)dt, \qquad (4.6)
\end{aligned}$$

where $B(0,t)=A_1(0,t)e^{-A_2(0,t)r_0}$, $A_1(0,t)=\exp\left(\left(\theta-\frac{\sigma^2}{2k^2}\right)[A_2(0,t)-t]-\frac{\sigma^2A_2^2(0,t)}{4k}\right)$ and $A_2(0,t)=\frac{1-e^{-tk}}{k}$.

Similarity, we apply Corollary 3.2.11 by substituting

$$\begin{aligned}
\frac{d\left(\ln A_1(s,t)\right)}{dt} =& \left(\theta-\frac{\sigma^2}{2k^2}\right)\left[e^{-k(t-s)}-1\right]-\frac{\sigma^2e^{-k(t-s)}\left(1-e^{-k(t-s)}\right)}{2k^2} \\
\frac{dA_2(s,t)}{dt} =& e^{-k(t-s)} \\
\frac{d\left(\ln A_1(0,t)\right)}{dt} =& \left(\theta-\frac{\sigma^2}{2k^2}\right)\left[e^{-kt}-1\right]-\frac{\sigma^2e^{-kt}\left(1-e^{-kt}\right)}{2k^2} \\
\frac{dA_2(0,t)}{dt} =& e^{-kt}.
\end{aligned}$$

into the following equation

$$\begin{aligned}
&\mathrm{E}\left[r_s e^{-\int_0^t r_v dv}\right] \\
=& \frac{\frac{d\ln(A_1(s,t))}{dt}-\frac{d\ln(A_1(0,t))}{dt}+r_0\frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}}B(0,t) \\
=& \frac{\left(\theta-\frac{\sigma^2}{2k^2}\right)\left[e^{-k(t-s)}-e^{-kt}\right]-\frac{\sigma^2}{2k^2}\left[e^{-k(t-s)}\left(1-e^{-k(t-s)}\right)-e^{-kt}\left(1-e^{-kt}\right)\right]+r_0e^{-kt}}{e^{-k(t-s)}}B(0,t) \\
=& \left[\left(\theta+(r_0-\theta)e^{-ks}\right)-\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\right]B(0,t),
\end{aligned}$$

which implies we can obtain the same formula for $\mathrm{E}[M_s]$ with (4.6).

Thus, the optimal time to refinance when $\rho = 1$, can be solved by the following equation with numerical methods.

$$\begin{aligned}
&\int_s^T \left[r_0 - \mathrm{E}[r_s] + \frac{\sigma^2}{k} \left(\frac{1-e^{-ks}}{k} - \frac{e^{-k(t-s)} - e^{-k(t+s)}}{2k} \right) \right] B(0,t)dt \\
=& -\left\{ \int_s^T \left[k(r_0 - \theta)e^{-ks} + \frac{\sigma^2}{k} \left(e^{-ks} - \frac{e^{-k(t-s)} + e^{-k(t+s)}}{2} \right) \right] B(0,t)dt \right. \\
&+ \left. \left[r_0 - \mathrm{E}[r_s] + \frac{\sigma^2}{k} \left(\frac{1-e^{-ks}}{k} - \frac{1-e^{-2ks}}{2k} \right) \right] B(0,s) \right\} \frac{(T-s)\left[e^{-r_0(T-s)} + r_0(T-s) - 1 \right]}{\left(r_0(T-s)+1 \right) e^{-r_0(T-s)} - 1}.
\end{aligned} \tag{4.7}$$

Figure 4.2 represents the numerical value of $\mathrm{E}[M_s]$ with different parameters. It is clearly that the maximum point appears at the early stage of the life of the contract. Moreover, k and σ are two important parameters who have significant impacts on the value of $\mathrm{E}[M_s]$ and the optimal time to refinance. The results show that the optimal time, with $\rho = 1$, will be shorter when we either increase k or σ^2. Furthermore, the numerical value of $\mathrm{E}[M_s]$ will increase with the increasing of σ^2, which can be proved based on (4.6). However, we may obtain an infinite value of $\mathrm{E}[M_s]$ if the parameters chosen for k, θ and σ^2 lead to the infinite value of $A_1(0,t)$.

Table A.2 displays the numerical result of (4.7). The optimal time to refinance, with $\rho = 1$, will increase with the increase of the life of the contract T. Moreover, one can see that if we let $s = f(T)$, the slope of $f(T)$ will decrease gradually when T increases. Eventually, the slope will be zero, say, when T is greater then 80, as described in Table A.2.

Lemma 4.2.2. *The numerical value of* $\mathrm{E}[M_s]$ *will increase with the increasing of* σ^2.

Proof. To evaluate the value of $\mathrm{E}[M_s]$ with respect to σ^2, we let

$$\begin{aligned}
f\left(\sigma^2\right) =& \left[r_0 - \left(\theta + (r_0 - \theta)e^{-ks} \right) + \frac{\sigma^2}{k} \left(\frac{1-e^{-ks}}{k} - \frac{e^{-k(t-s)} - e^{-k(t+s)}}{2k} \right) \right] B(0,t) \\
=& aB(0,t) + b\sigma^2 B(0,t) \\
=& ae^{c\sigma^2} e^{\theta[A_2(0,t)-t] - A_2(0,t)r_0} + b\sigma^2 e^{c\sigma^2} e^{\theta[A_2(0,t)-t] - A_2(0,t)r_0},
\end{aligned}$$

with

$$\begin{aligned}
a =& r_0 - \left(\theta + (r_0 - \theta)e^{-ks} \right) \\
b =& \frac{1-e^{-ks}}{k^2} - \frac{e^{-k(t-s)} - e^{-k(t+s)}}{2k^2} \\
c =& -\frac{A_2(0,t) - t}{2k^2} - \frac{A_2^2(0,t)}{4k}.
\end{aligned}$$

Taking the first derivative of $f\left(\sigma^2\right)$ with respect to σ^2 gives

$$\frac{df\left(\sigma^2\right)}{d\sigma^2} = ace^{c\sigma^2}e^{\theta[A_2(0,t)-t]-A_2(0,t)r_0} + be^{c\sigma^2}e^{\theta[A_2(0,t)-t]-A_2(0,t)r_0} + bc\sigma^2 e^{c\sigma^2}e^{\theta[A_2(0,t)-t]-A_2(0,t)r_0}.$$

As we can see that a,b and c are positive given $k > 0$ and $t > s > 0$, we have $\frac{df(\sigma^2)}{d\sigma^2} > 0$. Thus, $f\left(\sigma^2\right)$ is an increasing function with respect to σ^2. $\square$

Remark 4.2.2. *a,b and c in Lemma 4.2.2 are positive. As we know that $k > 0$ and $t > s > 0$, we have*

$$a = r_0 - \left(\theta + (r_0 - \theta)e^{-ks}\right) = (r_0 - \theta)\left(1 - e^{-ks}\right) > 0,$$

$$b = \frac{1 - e^{-ks}}{k^2} - \frac{e^{-k(t-s)} - e^{-k(t+s)}}{2k^2} = \frac{2 - 2e^{-ks} - e^{-k(t-s)} + e^{-k(t+s)}}{2k^2}.$$

We let $f(t) = 2 - 2e^{-ks} - e^{-k(t-s)} + e^{-k(t+s)}$, thus, $f'(t) = k\left(e^{-k(t-s)} - e^{-k(t+s)}\right)$. As we know that $k > 0$ and $t > s > 0$, thus $f'(t) > 0$, implying $b = \frac{f(t)}{2k^2} > \frac{f(s)}{2k^2} = \frac{1-2e^{-ks}+e^{-2ks}}{2k^2}$. As we have

$$f'(s) = 2k\left(e^{-ks} - e^{-2ks}\right) > 0,$$

thus, $f(s) > f(0) = 0$, implying that $b > 0$.

$$\begin{aligned} c = & -\frac{A_2(0,t) - t}{2k^2} - \frac{A_2^2(0,t)}{4k} \\ = & -\frac{1 - e^{-kt} - kt}{2k^3} - \frac{1 - 2e^{-kt} + e^{-2kt}}{4k^3} \\ = & \frac{4e^{-kt} - 3 + 2kt - e^{-2kt}}{4k^3}. \end{aligned}$$

We let $g(t) = 4e^{-kt} - 3 + 2kt - e^{-2kt}$, thus

$$\begin{aligned} g'(t) = & -4ke^{-kt} + 2k + 2ke^{-2kt} \\ g''(t) = & 4k^2e^{-kt} - 4k^2e^{-2kt} = 4k^2\left(e^{-kt} - e^{-2kt}\right). \end{aligned}$$

As $t > 0$ and $k > 0$, we have $g''(t) > 0$, implying $g'(t) > g'(0) = 0$. We can see that $g(t)$ is an increasing function, and thus, $c = \frac{g(t)}{4k^3} > \frac{g(0)}{4k^3} = 0$.

As $\mathrm{E}\left[M_s^2\right]$ is given by

$$\mathrm{E}\left[M_s^2\right] = 2\left(P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1 - e^{-r_0T}\right]}\right)^2 \int_s^T \int_s^t \mathrm{E}\left[(r_0 - r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right] dh dt,$$

thus, we compute the value of $\mathrm{E}\left[(r_0 - r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right]$ under Vasicek model as following

$$
\begin{aligned}
&\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] \\
=&\mathrm{E}\left[\left[(r_0-\theta)\left(1-e^{-ks}\right)-\sigma e^{-ks}\int_0^s e^{ku}dW_u\right]^2 e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\right] \\
=&e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}}\left\{(r_0-\theta)^2\left(1-e^{-ks}\right)^2\mathrm{E}\left[e^{-\frac{\sigma\int_0^h 1-e^{-k(h-v)}dW_v}{k}}e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right]\right. \\
&-2(r_0-\theta)\left(1-e^{-ks}\right)\mathrm{E}\left[\int_0^s \sigma e^{-k(s-v)}dW_v e^{-\frac{\sigma\int_0^h 1-e^{-k(h-v)}dW_v}{k}}e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right] \\
&\left.+\sigma^2\mathrm{E}\left[\left(\int_0^s e^{-k(s-v)}dW_v\right)^2 e^{-\frac{\sigma\int_0^h 1-e^{-k(h-v)}dW_v}{k}}e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right]\right\},
\end{aligned}
$$

We assume $f_1(v) = -\frac{\sigma}{k}\left[2-e^{-k(h-v)}-e^{-k(t-v)}\right]$, $g_1(v) = \sigma e^{-k(s-v)}$, and $q_1(v) = -\frac{\sigma}{k}\left[1-e^{-k(t-v)}\right]$, thus, we continue calculating $\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\Big]$ as

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\frac{\sigma\int_0^h 1-e^{-k(h-v)}dW_v}{k}}e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right] \\
=&\mathrm{E}\left[e^{\int_0^h f_1(v)dW_v}\right]\mathrm{E}\left[e^{\int_h^t q_1(v)dW_v}\right] \\
=&\exp\left[\frac{\sigma^2}{2k^2}\left(4h+4\frac{e^{-kt}+e^{-kh}}{k}-3\frac{e^{-k(t-h)}}{k}-\frac{7}{2k}-\frac{e^{-k(t+h)}}{k}+\frac{e^{-2k(t-h)}-e^{-2kt}-e^{-2kh}}{2k}\right)\right] \\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t-h+2\frac{e^{-k(t-h)}}{k}-\frac{e^{-2k(t-h)}}{2k}-\frac{3}{2k}\right)\right] \\
=&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^s \sigma e^{-k(s-v)}dW_v e^{-\frac{\sigma\int_0^h 1-e^{-k(t-v)}dW_v}{k}}e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right] \\
=&\mathrm{E}\left[\int_0^s g_1(v)dW_v e^{\int_0^s f_1(v)dW_v}\right]\mathrm{E}\left[e^{\int_s^h f_1(v)dW_v}\right]\mathrm{E}\left[e^{\int_h^t q_1(v)dW_v}\right] \\
=&\frac{d}{d\alpha}|_{\alpha=0}e^{\int_0^s \frac{(\alpha g(v)+f(v))^2}{2}dv}\mathrm{E}\left[e^{\int_s^h f_1(v)dW_v}\right]\mathrm{E}\left[e^{\int_h^t q_1(v)dW_v}\right] \\
=&\int_0^s g_1(v)f_1(v)dv e^{\int_0^h \frac{f_1^2(v)}{2}dv}e^{\int_h^t \frac{q_1^2(v)}{2}dv} \\
=&-\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right) \\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right],
\end{aligned}
$$

and

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_0^s \sigma e^{-k(s-v)}dW_v\right)^2 e^{-\frac{\sigma\int_0^h 1-e^{-k(h-v)}dW_v}{k}} e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right] \\
=&\mathrm{E}\left[\left(\int_0^s g_1(v)dW_v\right)^2 e^{\int_0^s f_1(v)dW_v}\right]\mathrm{E}\left[e^{\int_s^h f_1(v)dW_v}\right]\mathrm{E}\left[e^{\int_h^t q_1(v)dW_v}\right].
\end{aligned}$$

As we have

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_0^s g_1(v)dW_v\right)^2 e^{\int_0^s f_1(v)dW_v}\right] \\
=&\frac{d^2}{d\alpha^2}|_{\alpha=0} e^{\int_0^s \frac{(\alpha g_1(v)+f_1(v))^2}{2}dv} \\
=&\left[\int_0^s g_1^2(v)dv + \left[\int_0^s (\alpha g_1(v)+f_1(v))\, g_1(v)dv\right]^2\right] e^{\int_0^s \frac{[\alpha g_1(v)+f_1(v)]^2}{2}dv}|_{\alpha=0} \\
=&\left[\int_0^s g_1^2(v)dv + \left(\int_0^s f_1(v)g_1(v)dv\right)^2\right] e^{\int_0^s \frac{[f(v)]^2}{2}dv},
\end{aligned}$$

thus, we can obtain

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_0^s \sigma e^{-k(s-v)}dW_v\right)^2 e^{-\frac{\sigma\int_0^h 1-e^{-k(h-v)}dW_v}{k}} e^{-\frac{\sigma\int_0^t 1-e^{-k(t-v)}dW_v}{k}}\right] \\
=&\left[\int_0^s g_1^2(v)dv + \left(\int_0^s f_1(v)g_1(v)dv\right)^2\right] e^{\int_0^h \frac{[f_1(v)]^2}{2}dv} e^{\int_h^t \frac{[q_1(v)]^2}{2}dv} \\
=&\left[\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right) + \frac{\sigma^4}{4k^4}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)^2\right] \\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k} - \frac{e^{-k(t-h)}+e^{-k(t+h)}}{k} - \frac{5}{k} - \frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathrm{E}\left[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right] \\
=&e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}}\Bigg\{(r_0-\theta)^2\left(1-e^{-ks}\right)^2 \\
&+2(r_0-\theta)\left(1-e^{-ks}\right)\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right) \\
&+\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right)+\frac{\sigma^4}{4k^4}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)^2\Bigg\} \\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k} - \frac{e^{-k(t-h)}+e^{-k(t+h)}}{k} - \frac{5}{k} - \frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right].
\end{aligned}$$

Similarity, we can apply Lemma 3.2.16, Corollary 3.2.18 and 3.2.19 to Vasicek model. The SDE of $\hat{r}_t$ is

$$d\hat{r}_t = \hat{k}\left(\hat{\theta}-\hat{r}_t\right)dt + \hat{\sigma}dW_t,$$

thus, we have $\hat{k}=k$, $\hat{\theta}=2\theta$ and $\hat{\sigma}=2\sigma$. The bond price under $\hat{r}_t$ is

$$\hat{B}(h,t)=\hat{A}_1(h,t)e^{-\hat{A}_2(h,t)\hat{r}_h},$$

with

$$\begin{aligned}
\hat{A}_1(h,t)=&\exp\left(\left(\hat{\theta}-\frac{\hat{\sigma}^2}{2\hat{k}^2}\right)[\hat{A}_2(h,t)-(t-h)]-\frac{\hat{\sigma}^2\hat{A}_2^2(h,t)}{4\hat{k}}\right)\\
=&\exp\left(\left(2\theta-\frac{4\sigma^2}{2k^2}\right)[\hat{A}_2(h,t)-(t-h)]-\frac{4\sigma^2\hat{A}_2^2(h,t)}{4k}\right)\\
\hat{A}_2(h,t)=&\frac{1-e^{-(t-h)k}}{k}.
\end{aligned}$$

As $\frac{A_2(h,t)}{2}=\hat{A}_2(h,\tilde{t})$ gives $\frac{1-e^{-\tilde{t}k}}{k}=\frac{2-e^{-tk}-e^{-hk}}{2k}$ based on Lemma 3.2.16, we have

$$\begin{aligned}
&\mathrm{E}\left[e^{-\int_0^h r_v dv}e^{-\int_0^t r_v dv}\right]\\
=&\frac{A_1(h,t)\hat{A}_1(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}e^{-\hat{A}_2(0,\tilde{t})\hat{r}_0}\\
=&\frac{\exp\left(\left(\theta-\frac{\sigma^2}{2k^2}\right)[A_2(h,t)-(t-h)]-\frac{\sigma^2A_2^2(h,t)}{4k}+\left(2\theta-\frac{4\sigma^2}{2k^2}\right)[\hat{A}_2(0,\tilde{t})-\tilde{t}]-\frac{4\sigma^2\hat{A}_2^2(0,\tilde{t})}{4k}\right)}{\exp\left(\left(2\theta-\frac{4\sigma^2}{2k^2}\right)[\hat{A}_2(h,\tilde{t})-(\tilde{t}-h)]-\frac{4\sigma^2\hat{A}_2^2(h,\tilde{t})}{4k}\right)}e^{\frac{1-e^{-\tilde{t}k}}{k}\hat{r}_0}\\
=&\frac{\exp\left(\left(\theta-\frac{\sigma^2}{2k^2}\right)[A_2(h,t)-(t-h)]-\frac{\sigma^2A_2^2(h,t)}{4k}+\left(2\theta-\frac{4\sigma^2}{2k^2}\right)[\hat{A}_2(0,\tilde{t})-\tilde{t}]-\frac{4\sigma^2\hat{A}_2^2(0,\tilde{t})}{4k}\right)}{\exp\left(\left(2\theta-\frac{4\sigma^2}{2k^2}\right)\left[\frac{A_2(h,t)}{2}-(\tilde{t}-h)\right]-\frac{4\sigma^2\left(\frac{A_2(h,t)}{2}\right)^2}{4k}\right)}e^{\frac{1-e^{-\tilde{t}k}}{k}\hat{r}_0}\\
=&\exp\left(-\theta t-\theta h+\frac{\sigma^2}{2k^2}A_2(h,t)+\frac{\sigma^2}{2k^2}t+\frac{3\sigma^2}{2k^2}h+2\theta\hat{A}_2(0,\tilde{t})-\frac{4\sigma^2}{2k^2}\hat{A}_2(0,\tilde{t})-\frac{\sigma^2}{k}\hat{A}_2^2(0,\tilde{t})\right)e^{\frac{1-e^{-\tilde{t}k}}{k}\hat{r}_0}\\
=&\exp\left(-\theta(t+h)+2\theta\hat{A}_2(0,\tilde{t})+\frac{\sigma^2}{2k^2}\left(A_2(h,t)+t+3h-4\hat{A}_2(0,\tilde{t})-2k\hat{A}_2^2(0,\tilde{t})\right)\right)e^{-r_0\frac{2-e^{-kh}-e^{-kt}}{k}}\\
=&e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}}\\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right].
\end{aligned}$$

We apply Corollary 3.2.18 to Vasicek model, which gives

$$
\begin{aligned}
&\mathrm{E}\left[r_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right] \\
=&\frac{1}{2}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}+\hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t}) \\
=&\frac{1}{2}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[\frac{2\theta\left(e^{-k\left(\tilde{t}-s\right)}-e^{-k\tilde{t}}\right)-\frac{2\sigma^2}{k^2}\left(e^{-k\left(\tilde{t}-s\right)}-e^{-k\tilde{t}}\right)}{e^{-k\tilde{t}}}\right. \\
&-\left.\frac{\frac{2\sigma^2}{k^2}\left(1-e^{-k\left(\tilde{t}-s\right)}\right)e^{-k\left(\tilde{t}-s\right)}-\frac{2\sigma^2}{k^2}\left(1-e^{-k\tilde{t}}\right)e^{-k\tilde{t}}}{e^{-k\tilde{t}}}\right] \\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[\theta\left(1-e^{-ks}\right)-\frac{\sigma^2}{k^2}\left(1-e^{-ks}\right)-\frac{\sigma^2}{k^2}\left(1-e^{-k\left(\tilde{t}-s\right)}\right)\right. \\
&+\left.\frac{\sigma^2}{k^2}\left(1-e^{-k\tilde{t}}\right)e^{-ks}+r_0e^{-ks}\right] \\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[r_0e^{-ks}+\theta\left(1-e^{-ks}\right)-\frac{\sigma^2}{k^2}\left(2-2e^{-ks}-e^{-k\left(\tilde{t}-s\right)}+e^{-k\left(\tilde{t}+s\right)}\right)\right] \\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[r_0e^{-ks}+\theta\left(1-e^{-ks}\right)-\frac{\sigma^2}{k^2}\left(2-2e^{-ks}+\frac{e^{-kt}+e^{-kh}}{2}e^{-ks}-\frac{e^{-kt}+e^{-kh}}{2}e^{ks}\right)\right] \\
=&\left[r_0e^{-ks}+\theta\left(1-e^{-ks}\right)-\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)\right] \\
&e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}} \\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right].
\end{aligned}
$$

To simplify our notation, we let

$$
X=-\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right),
$$

hence, base on Corollary 3.2.19, we have

$$
\begin{aligned}
&\mathrm{E}\left[r_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right] \\
=&\frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}}-\frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}}+\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2\right.\\
-&\left.\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0-\frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right]\hat{B}(0,\tilde{t})\\
+&\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0+\frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}+\hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t})\\
=&\frac{1}{4}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})e^{-2k(\tilde{t}-s)}}\left[\left(-2k\theta+\frac{2\sigma^2}{k}\right)\left(e^{-k\tilde{t}}-e^{-k(\tilde{t}-s)}\right)\right.\\
+&\left.\frac{2\sigma^2}{k}\left(e^{-k\tilde{t}}-e^{-2k\tilde{t}}-e^{-k(\tilde{t}-s)}+e^{-2k(\tilde{t}-s)}\right)\right]\\
+&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[\theta^2\left(1-e^{-ks}\right)^2+2\theta\left(1-e^{-ks}\right)X+X^2+\frac{1}{2}\left(\theta\left(1-e^{-ks}\right)+X\right)e^{k(\tilde{t}-s)}\hat{r}_0e^{-k\tilde{t}}\right.\\
+&\left.\frac{1}{2}r_0ke^{k\tilde{t}}e^{2ks}\right]+\frac{1}{2}\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[e^{-k\tilde{t}}\hat{r}_0-k\right]e^{k(\tilde{t}-s)}\left(r_0e^{-ks}+\theta\left(1-e^{-ks}\right)+X\right)\\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[r_0^2e^{-2ks}+2r_0\theta\left(1-e^{-ks}\right)e^{-ks}+\theta^2\left(1-e^{-ks}\right)^2+2r_0e^{-ks}X+2\theta\left(1-e^{-ks}\right)X\right.\\
+&\left.\frac{\sigma^2}{2k}\left(2e^{-k\tilde{t}}-2e^{-2k\tilde{t}}-2e^{-k(\tilde{t}-s)}+2e^{-2k(\tilde{t}-s)}\right)e^{2k(\tilde{t}-s)}-\frac{1}{2}kXe^{k(\tilde{t}-s)}\right]\\
=&\left[r_0^2e^{-2ks}+2r_0\theta\left(1-e^{-ks}\right)e^{-ks}+\theta^2\left(1-e^{-ks}\right)^2+2r_0e^{-ks}X+2\theta\left(1-e^{-ks}\right)X+X^2\right.\\
+&\left.X^2+\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right)\right]e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}}\\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big]\\
=&r_0^2\mathrm{E}\Big[e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] - 2r_0\mathrm{E}\Big[r_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big] + \mathrm{E}\Big[r_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big]\\
=&e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}}\Bigg\{(r_0-\theta)^2\left(1-e^{-ks}\right)^2\\
+&2(r_0-\theta)\left(1-e^{-ks}\right)\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)\\
+&\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right)+\frac{\sigma^4}{4k^4}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)^2\Bigg\}\\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right].
\end{aligned}
$$

The variance of M_s is

$$
\begin{aligned}
&\mathrm{V}ar[M_s]\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t \mathrm{E}\Big[(r_0-r_s)^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\Big]dhdt-(\mathrm{E}[M_s])^2\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t e^{-\frac{(r_0-\theta)\left(2-e^{-kt}-e^{-kh}\right)+k\theta(t+h)}{k}}\Bigg\{(r_0-\theta)^2\left(1-e^{-ks}\right)^2\\
+&2(r_0-\theta)\left(1-e^{-ks}\right)\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)\\
+&\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right)+\frac{\sigma^4}{4k^4}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)^2\Bigg\}\\
&\exp\left[\frac{\sigma^2}{2k^2}\left(t+3h+4\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{5}{k}-\frac{e^{-2kt}+e^{-2kh}}{2k}\right)\right]dhdt-(\mathrm{E}[M_s])^2\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t\Bigg\{(r_0-\theta)^2\left(1-e^{-ks}\right)^2\\
+&2(r_0-\theta)\left(1-e^{-ks}\right)\frac{\sigma^2}{2k^2}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)\\
+&\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right)+\frac{\sigma^4}{4k^4}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)^2\Bigg\}\\
&\exp\left[\frac{\sigma^2}{2k^2}\left(2h+2\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{2}{k}\right)\right]B(0,h)B(0,t)dhdt-(\mathrm{E}[M_s])^2\,,
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
[\mathrm{E}[M_s]]^2 =& \left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\left(\int_s^T\left[(r_0-\theta)\left(1-e^{-ks}\right)\right.\right.\\
&\left.\left.+\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\right]B(0,t)dt\right)^2\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t\left(\left[(r_0-\theta)\left(1-e^{-ks}\right)\right.\right.\\
&\left.\left.+\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\right]B(0,t)\right)\left(\left[(r_0-\theta)\left(1-e^{-ks}\right)\right.\right.\\
&\left.\left.+\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(h-s)}-e^{-k(h+s)}}{2k}\right)\right]B(0,h)\right)dhdt\\
=&2\left(P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\right)^2\int_s^T\int_s^t B(0,h)B(0,t)\left[(r_0-\theta)^2\left(1-e^{-ks}\right)^2\right.\\
&+(r_0-\theta)\left(1-e^{-ks}\right)\frac{\sigma^2}{2k^2}\left(4-4e^{-ks}-e^{-k(t-s)}+e^{-k(t+s)}-e^{-k(h-s)}+e^{-k(h+s)}\right)\\
&+\frac{\sigma^4}{4k^4}\left(4-8e^{-ks}-2e^{-k(h-s)}+2e^{-k(h+s)}+4e^{-2ks}+2e^{-kh}-2e^{-k(h+2s)}-2e^{-k(t-s)}\right.\\
&\left.\left.+2e^{-kt}+e^{-k(t+h-2s)}-2e^{-k(t+h)}+2e^{-k(t+s)}-2e^{-k(t+2s)}+e^{-k(t+h+2s)}\right)\right]dhdt.
\end{aligned}
$$

Figure 4.3 displays the value of $\mathrm{V}ar[M_s]$ in (4.8). As shown in Figure 4.3, the time where $\mathrm{V}ar[M_s]$ reaches the maximum is less or equal to 5 years. We apply the utility function presented in (3.7) to Vasicek model. As in our problem, we assume $\mathrm{E}[M_s]$ will dominate the utility function, we may let $\rho \in (0.5, 1]$. Figure 4.4 shows the value of the utility function under Vasicek model. It can be seen that the optimal refinancing time, obtained by the optimization of the utility function will decrease with the increase of k. The results are reasonable as increase k will enhance the probability of lower interest rate, governed by the Vasicek model. From Table A.3, one can see that the optimal time will decrease when ρ increases. With the increasing of ρ, $\mathrm{V}ar[M_s]$ will have less impact on the utility function, resulting in the decreasing of the optimal time. In particular, when $\rho = 1$, the optimal time will be obtained by (4.7). In this case, $\mathrm{E}[M_s]$ will be taken into consideration in the utility function, implying the optimal refinancing time will be shortest.

4.2.3 $T = \infty$

The value of $\mathrm{E}[M_s]$ can be evaluated as followings with $T = \infty$.

$$
\mathrm{E}[M_s] = P(0)\int_s^\infty\left[r_0-\mathrm{E}[r_s]+\frac{\sigma^2}{k}\left(\frac{1-e^{-ks}}{k}-\frac{e^{-k(t-s)}-e^{-k(t+s)}}{2k}\right)\right]B(0,t)dt. \tag{4.9}
$$

As one can see from (4.9), the stableness of $\mathrm{E}[M_s]$ solely depends on the stableness of $B(0,t)$ under Vasicek model. And the variance of M_s is

$$
\begin{aligned}
&\mathrm{V}ar[M_s]\\
=&2P(0)^2\int_s^\infty\int_s^t\\
&\Bigg\{(r_0-\theta)^2\left(1-e^{-ks}\right)^2\left[\exp\left[\frac{\sigma^2}{2k^2}\left(2h+2\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{2}{k}\right)\right]-1\right]\\
&+(r_0-\theta)\left(1-e^{-ks}\right)\frac{\sigma^2}{2k^2}\left(4-4e^{-ks}-e^{-k(t-s)}+e^{-k(t+s)}-e^{-k(h-s)}+e^{-k(h+s)}\right)\\
&\left[2\exp\left[\frac{\sigma^2}{2k^2}\left(2h+2\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{2}{k}\right)\right]-1\right]\\
&+\left(\frac{\sigma^2}{2k}\left(1-e^{-2ks}\right)+\frac{\sigma^4}{4k^4}\left(4+e^{-k(t+s)}+e^{-k(h+s)}-e^{-k(t-s)}-e^{-k(h-s)}-4e^{-ks}\right)^2\right)\\
&\exp\left[\frac{\sigma^2}{2k^2}\left(2h+2\frac{e^{-kt}+e^{-kh}}{k}-\frac{e^{-k(t-h)}+e^{-k(t+h)}}{k}-\frac{2}{k}\right)\right]\\
&-\frac{\sigma^4}{4k^4}\Bigg(4-8e^{-ks}-2e^{-k(h-s)}+2e^{-k(h+s)}+4e^{-2ks}+2e^{-kh}-2e^{-k(h+2s)}-2e^{-k(t-s)}\\
&+2e^{-kt}+e^{-k(t+h-2s)}-2e^{-k(t+h)}+2e^{-k(t+s)}-2e^{-k(t+2s)}+e^{-k(t+h+2s)}\Bigg)\Bigg\}B(0,h)B(0,t)dhdt.
\end{aligned}
$$

4.3 CIR Model

The CIR short term interest rate process, first proposed by Cox, is a mathematical model describing the evolution of interest rate. The model specifies that under the risk-neutral measure Q, the instantaneous interest rate follows the stochastic differential equation

$$dr_t = k\left(\theta - r_t\right)dt + \sigma\sqrt{r_t}dW_t. \tag{4.10}$$

It is well-known that $B(s,t)$ described in (3.2) has the analytic solution when r_t follows (4.10), which gives

$$B(s,t) = A_1(s,t)\exp\left(-A_2(s,t)r_s\right),$$

where

$$
\begin{aligned}
A_1(s,t) =&\left(\frac{2\omega e^{\frac{(k+\omega)(t-s)}{2}}}{2\omega+(k+\omega)\left[e^{(t-s)\omega}-1\right]}\right)^{\frac{2k\theta}{\sigma^2}}\\
A_2(s,t) =&\frac{2\left[e^{(t-s)\omega}-1\right]}{2\omega+(k+\omega)\left[e^{(t-s)\omega}-1\right]}\\
\omega =&\sqrt{k^2+2\sigma^2}.
\end{aligned}
$$

4.3.1 Preliminary analysis

Suppose at time s (in the following graph 4.5, we suppose s is the date the debtor would like to consider refinancing) . Intuitively, one would possibly refinance at s only if $c_s < c_0$, although this statement may be slightly challenged by a debtor who argues to keep waiting, betting on a even better deal in future. To heuristically illustrate, we display, in the following Figure. 4.5, the comparative level plots of initial mortgage rate c_0 and the mortgage rate process. For convenience, all these plots are based on the assumption that $\sigma = 0$.

1. If $c_0 < c_s$, then for this scenario, there is zero possibility for the debtor to optimally refinance at s. If he or she refinances, he or she immediately pays higher monthly instalment, giving up the existing lower interest and also the possibility of future refinancing where the interest rate can be better a deal depending on whether the market trend goes deeply down enough.

2. If $c_0 > c_s$ and c_s is set to climb in future trend, these are typical scenarios where the debtor would possibly refinance today. If he or she does so, he or she immediately enjoys a lower interest and a lower monthly payment. The longer he or she keeps waiting, the higher the interest rate. Also, the longer he or she waits, the lower face value of the original loan, and the less benefit of refinancing. This observation has been numerically verified in our previous paper with plots of the density functions of optimal refinancing time, showing that the optimal refinancing, if exists, usually occurs at the early stage of the contract. But why does not the debtor always refinance at s for this case? It is because of the market volatility. The market volatility issues a small probability that to wait for a little while further actually grants even better deals.

3. If $c_0 > c_s$ and c_s is strictly decreasing, even if $\sigma = 0$ for this case (a higher $\sigma > 0$ is usually the main reason for debtor to take a wait-and-see strategy), chances are debtor can wait for a while to optimally refinance. How long to wait depends on how fast and how low the interest goes down in future. This is the most interesting case to which our method and implementation are dedicated in the subsequent analysis.

4.3.2 $T < \infty$

As the explicit formula for r_s is not given under CIR model, we may apply Theorem 3.2.2 to obtain $\mathrm{E}[M_s]$ as

$$\mathrm{E}[M_s] = P(0)\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1 - e^{-r_0 T}\right]}\int_s^T \left[r_0 - \frac{\frac{d\ln(A_1(s,t))}{dt} - \frac{d\ln(A_1(0,t))}{dt} + r_0\frac{dA_2(0,t)}{dt}}{\frac{dA_2(s,t)}{dt}}\right] B(0,t)dt,$$

where

$$
\begin{aligned}
\frac{d\left(\ln A_1(s,t)\right)}{dt} =& \frac{k\theta}{\sigma^2}\left[k+\omega-\frac{2\omega(k+\omega)e^{(t-s)\omega}}{2\omega+(k+\omega)\left[e^{(t-s)\omega}-1\right]}\right] \\
\frac{dA_2(s,t)}{dt} =& \frac{2\omega e^{(t-s)\omega}}{2\omega+(k+\omega)\left[e^{(t-s)\omega}-1\right]}-\frac{2\omega(k+\omega)e^{(t-s)\omega}\left[e^{(t-s)\omega}-1\right]}{\left[2\omega+(k+\omega)\left[e^{(t-s)\omega}-1\right]\right]^2} \\
=& \frac{4\omega^2 e^{(t-s)\omega}}{\left[2\omega+(k+\omega)\left[e^{(t-s)\omega}-1\right]\right]^2} \\
\frac{d\left(\ln A_1(0,t)\right)}{dt} =& \frac{k\theta}{\sigma^2}\left[k+\omega-\frac{2\omega(k+\omega)e^{t\omega}}{2\omega+(k+\omega)\left[e^{t\omega}-1\right]}\right] \\
\frac{dA_2(0,t)}{dt} =& \frac{2\omega e^{t\omega}}{2\omega+(k+\omega)\left[e^{t\omega}-1\right]}-\frac{2\omega(k+\omega)e^{t\omega}\left[e^{t\omega}-1\right]}{\left[2\omega+(k+\omega)\left[e^{t\omega}-1\right]\right]^2} \\
=& \frac{4\omega^2 e^{t\omega}}{\left[2\omega+(k+\omega)\left[e^{t\omega}-1\right]\right]^2},
\end{aligned}
$$

and thus, we have

$$
\begin{aligned}
&\mathrm{E}[M_s] \qquad (4.11)\\
&= P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}\int_s^T\left[r_0-r_0e^{s\omega}\frac{\left[2\omega+(k+\omega)\left(e^{(t-s)\omega}-1\right)\right]^2}{\left[2\omega+(k+\omega)\left(e^{t\omega}-1\right)\right]^2}\right. \\
&\left.-k\theta\frac{\left(e^{s\omega}-1\right)\left[2\omega+(k+\omega)\left(e^{(t-s)\omega}-1\right)\right]}{\omega\left[2\omega+(k+\omega)\left(e^{t\omega}-1\right)\right]}\right]B(0,t)dt \\
&= P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]} \\
&\int_s^T\frac{(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)\left(1-e^{s\omega}\right)+(\omega+k)^2e^{2t\omega}\left(r_0-\frac{k\theta}{\omega}\right)\left(1-e^{-s\omega}\right)-\frac{2k\theta\sigma^2}{\omega}e^{t\omega}\left(e^{s\omega}-e^{-s\omega}\right)}{\left[2\omega+(k+\omega)\left(e^{t\omega}-1\right)\right]^2}B(0,t)dt.
\end{aligned}
$$

Hence, s can be obtained by the following equation with numerical methods

$$
\begin{aligned}
&\frac{\left(r_0(T-s)+1\right)e^{-r_0(T-s)}-1}{(T-s)\left[e^{-r_0(T-s)}+r_0(T-s)-1\right]} \qquad (4.12)\\
&\int_s^T\frac{(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)\left(1-e^{s\omega}\right)+(\omega+k)^2e^{2t\omega}\left(r_0-\frac{k\theta}{\omega}\right)\left(1-e^{-s\omega}\right)-\frac{2k\theta\sigma^2}{\omega}e^{t\omega}\left(e^{s\omega}-e^{-s\omega}\right)}{\left[2\omega+(k+\omega)\left(e^{t\omega}-1\right)\right]^2}B(0,t)dt \\
&=-\int_s^T\frac{-\omega(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)e^{s\omega}+\omega(\omega+k)^2e^{2t\omega}\left(r_0-\frac{k\theta}{\omega}\right)e^{-s\omega}-2k\theta\sigma^2e^{t\omega}\left(e^{s\omega}+e^{-s\omega}\right)}{\left[2\omega+(k+\omega)\left(e^{t\omega}-1\right)\right]^2}B(0,t)dt \\
&+\frac{(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)\left(1-e^{s\omega}\right)+(\omega+k)^2e^{2s\omega}\left(r_0-\frac{k\theta}{\omega}\right)\left(1-e^{-s\omega}\right)-\frac{2k\theta\sigma^2}{\omega}\left(e^{2s\omega}-1\right)}{\left[2\omega+(k+\omega)\left(e^{s\omega}-1\right)\right]^2}B(0,s).
\end{aligned}
$$

Figure 4.6 demonstrates the numerical value of $\mathrm{E}[M_s]$ based on CIR model. The result suggests refinancing should be considered within 5 years after signed the original contract. The graphs show that the optimal time to refinance, with $\rho = 1$, will be shorter when we increase k, which is consistent with the result in the Vasicek model.

Figure 4.7 describes the numerical value of $\mathrm{E}[M_s]$ through different approximating methods under CIR model. In Figure 4.7, 'over' represents the value of $\mathrm{E}[M_s]$ based on (3.15), while 'real' is the value based on (3.16). It is clearly that the approximation method based on (3.15) overstates the real value of $\mathrm{E}[M_s]$, which will benefit to the debtors. The result is consistent with the analytic discussion in Lemma 3.2.8.

The numerical solutions of the optimal time s, is presented in Table A.4. Similarly, the optimal time will increase when T increases, arriving at the stable point. Moreover, Figures 4.8, 4.9 and 4.10 display the relationship between the optimal time, k and σ^2. It is clearly that the optimal time will decrease with the increase of k or σ^2, as in CIR model, increasing k or σ^2 will enhance the probability of the lower interest rate, which shows the some property with the Vasicek model.

To apply Lemma 3.2.16, Corollary 3.2.18 and 3.2.19 to CIR model, we first describe the SDE of $\hat{r}_t$, which gives

$$d\hat{r}_t = \hat{k}\left(\hat{\theta} - \hat{r}_t\right) dt + \hat{\sigma}\sqrt{\hat{r}_t} dW_t,$$

thus, we have $\hat{k} = k$, $\hat{\theta} = 2\theta$ and $\hat{\sigma} = 2\sqrt{\sigma}$. The bond price under $\hat{r}_t$ is

$$\hat{B}(h,t) = \hat{A}_1(h,t)e^{-\hat{A}_2(h,t)\hat{r}_h},$$

with

$$\begin{aligned}
\hat{A}_1(h,t) &= \left(\frac{2\hat{\omega}e^{\frac{(\hat{k}+\hat{\omega})(t-h)}{2}}}{2\hat{\omega} + (\hat{k}+\hat{\omega})\left[e^{(t-h)\hat{\omega}} - 1\right]}\right)^{\frac{2\hat{k}\hat{\theta}}{\hat{\sigma}^2}} = \left(\frac{2\hat{\omega}e^{\frac{(k+\hat{\omega})(t-h)}{2}}}{2\hat{\omega} + (k+\hat{\omega})\left[e^{(t-h)\hat{\omega}} - 1\right]}\right)^{\frac{2k\theta}{\sigma^2}} \\
\hat{A}_2(h,t) &= \frac{2\left[e^{(t-h)\hat{\omega}} - 1\right]}{2\hat{\omega} + (k+\hat{\omega})\left[e^{(t-h)\hat{\omega}} - 1\right]} \\
\hat{\omega} &= \sqrt{k^2 + 4\sigma^2} = \sqrt{\omega^2 + 2\sigma^2}.
\end{aligned}$$

As $\frac{A_2(h,t)}{2} = \hat{A}_2(h,\tilde{t})$ gives $\frac{\left[e^{(t-h)\omega}-1\right]}{2\omega+(k+\omega)\left[e^{(t-h)\omega}-1\right]} = \frac{2\left[e^{(\tilde{t}-h)\hat{\omega}}-1\right]}{2\hat{\omega}+(k+\hat{\omega})\left[e^{(\tilde{t}-h)\hat{\omega}}-1\right]}$, implying

$$e^{\tilde{t}\hat{\omega}} = \left(\frac{\hat{\omega}A_2(h,t)}{2 - (k+\hat{\omega})\frac{A_2(h,t)}{2}} + 1\right)e^{h\hat{\omega}}.$$

V$ar[M_s]$ can obtained by (3.21), where

$$
\begin{aligned}
&\tilde{G}(0,s,\tilde{t})\\
=&\mathrm{E}\left[e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right]\\
=&\frac{A_1(h,t)\hat{A}_1(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})} e^{-\hat{A}_2(0,\tilde{t})\hat{r}_0}\\
=&\left(\frac{\frac{2\omega e^{\frac{(k+\omega)(t-h)}{2}}}{2\omega+(k+\omega)\left[e^{(t-h)\omega}-1\right]}}{\frac{2\hat{\omega} e^{\frac{(k+\hat{\omega})(\tilde{t}-h)}{2}}}{2\hat{\omega}+(k+\hat{\omega})\left[e^{(\tilde{t}-h)\hat{\omega}}-1\right]}} \frac{2\hat{\omega} e^{\frac{(k+\hat{\omega})\tilde{t}}{2}}}{2\hat{\omega}+(k+\hat{\omega})\left[e^{\tilde{t}\hat{\omega}}-1\right]}\right)^{\frac{2k\theta}{\sigma^2}} e^{-\frac{2\left[e^{\tilde{t}\hat{\omega}}-1\right]}{2\hat{\omega}+(k+\hat{\omega})\left[e^{\tilde{t}\hat{\omega}}-1\right]}\hat{r}_0}\\
=&\left(\frac{2\omega e^{\frac{(k+\omega)(t-h)}{2}}\left[e^{(\tilde{t}-h)\hat{\omega}}-1\right]}{\hat{\omega} e^{\frac{(k+\hat{\omega})(\tilde{t}-h)}{2}}\left[e^{(t-h)\omega}-1\right]} \frac{2\hat{\omega} e^{\frac{(k+\hat{\omega})\tilde{t}}{2}}}{2\hat{\omega}+(k+\hat{\omega})\left[e^{\tilde{t}\hat{\omega}}-1\right]}\right)^{\frac{2k\theta}{\sigma^2}} e^{-\frac{2\left[e^{\tilde{t}\hat{\omega}}-1\right]}{2\hat{\omega}+(k+\hat{\omega})\left[e^{\tilde{t}\hat{\omega}}-1\right]}\hat{r}_0}\\
=&\left(\frac{4\omega e^{\frac{(k+\omega)(t-h)}{2}} e^{\frac{(k+\hat{\omega})h}{2}}}{e^{(t-h)\omega}-1} \frac{e^{(\tilde{t}-h)\hat{\omega}}-1}{2\hat{\omega}+(k+\hat{\omega})\left[e^{\tilde{t}\hat{\omega}}-1\right]}\right)^{\frac{2k\theta}{\sigma^2}} e^{-\frac{2\left[e^{\tilde{t}\hat{\omega}}-1\right]}{2\hat{\omega}+(k+\hat{\omega})\left[e^{\tilde{t}\hat{\omega}}-1\right]}\hat{r}_0}\\
=&\left(\frac{4\omega e^{\frac{(k+\omega)(t-h)}{2}} e^{\frac{(k+\hat{\omega})h}{2}} \frac{\hat{\omega}A_2(h,t)}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}}{\left[e^{(t-h)\omega}-1\right]\left[2\hat{\omega}+(k+\hat{\omega})\left[\frac{\hat{\omega}A_2(h,t)e^{h\hat{\omega}}}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}+e^{h\hat{\omega}}-1\right]\right]}\right)^{\frac{2k\theta}{\sigma^2}} e^{-\frac{4\left[\frac{\hat{\omega}A_2(h,t)e^{h\hat{\omega}}}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}+e^{h\hat{\omega}}-1\right]}{2\hat{\omega}+(k+\hat{\omega})\left[\frac{\hat{\omega}A_2(h,t)e^{h\hat{\omega}}}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}+e^{h\hat{\omega}}-1\right]}r_0},
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{G}_{\alpha}^{(1)}(0,s,\tilde{t})\\
=&\mathrm{E}\left[r_s e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right]\\
=&\frac{1}{2}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}+\hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t})\\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{2\hat{A}_1(h,\tilde{t})}\left[e^{s\hat{\omega}}\frac{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^2}{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^2}\hat{r}_0+\hat{k}\hat{\theta}\frac{\left(e^{s\hat{\omega}}-1\right)\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]}{\hat{\omega}\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]}\right]\\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[e^{s\hat{\omega}}\frac{\left[2\hat{\omega}+(k+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^2}{\left[2\hat{\omega}+(k+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^2}r_0+k\theta\frac{\left(e^{s\hat{\omega}}-1\right)\left[2\hat{\omega}+(k+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]}{\hat{\omega}\left[2\hat{\omega}+(k+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]}\right]\\
=&\frac{A_1(h,t)\hat{B}(0,\tilde{t})}{\hat{A}_1(h,\tilde{t})}\left[2\hat{\omega}+(k+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]\\
&\frac{(\hat{\omega}-k)\left(r_0e^{s\hat{\omega}}+\frac{k\theta\left(e^{s\hat{\omega}}-1\right)}{\hat{\omega}}\right)+(\hat{\omega}+k)e^{\tilde{t}\hat{\omega}}\left(r_0+\frac{k\theta\left(e^{s\hat{\omega}}-1\right)}{\hat{\omega}}\right)}{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^2}\\
=&\tilde{G}(0,s,\tilde{t})\left[2\hat{\omega}+(k+\hat{\omega})\left[\frac{\hat{\omega}A_2(h,t)e^{(h-s)\hat{\omega}}}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}+e^{(h-s)\hat{\omega}}-1\right]\right]\\
&\frac{(\hat{\omega}-k)\left(r_0e^{s\hat{\omega}}+\frac{k\theta\left(e^{s\hat{\omega}}-1\right)}{\hat{\omega}}\right)+(\hat{\omega}+k)\left(\frac{\hat{\omega}A_2(h,t)}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}+1\right)e^{h\hat{\omega}}\left(r_0+\frac{k\theta\left(e^{s\hat{\omega}}-1\right)}{\hat{\omega}}\right)}{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(\frac{\hat{\omega}A_2(h,t)e^{h\hat{\omega}}}{2-(k+\hat{\omega})\frac{A_2(h,t)}{2}}+e^{h\hat{\omega}}-1\right)\right]^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{G}_{\alpha}^{(2)}(0,s,\tilde{t})\\
=&\mathrm{E}\left[r_s^2 e^{-\int_0^h r_v dv} e^{-\int_0^t r_v dv}\right]\\
=&\frac{1}{4}\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\left(\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}\right)^2}\left[\frac{d^2\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d^2\tilde{t}}-\frac{d^2\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d^2\tilde{t}}+\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)^2\right.\\
&\left.-\left(\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}\right)\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0-\frac{d^2\hat{A}_2(0,\tilde{t})}{d\tilde{t}^2}\hat{r}_0\right]\hat{B}(0,\tilde{t})\\
&+\frac{1}{4}\left[\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}\hat{r}_0+\frac{\frac{d^2\hat{A}_2(s,\tilde{t})}{d\tilde{t}^2}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\right]\frac{A_1(h,t)}{\hat{A}_1(h,\tilde{t})\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\frac{\frac{d\ln\left(\hat{A}_1(s,\tilde{t})\right)}{d\tilde{t}}-\frac{d\ln\left(\hat{A}_1(0,\tilde{t})\right)}{d\tilde{t}}+\hat{r}_0\frac{d\hat{A}_2(0,\tilde{t})}{d\tilde{t}}}{\frac{d\hat{A}_2(s,\tilde{t})}{d\tilde{t}}}\hat{B}(0,\tilde{t})\\
=&\frac{1}{4}\tilde{G}(0,s,\tilde{t})\left[\frac{k\theta\left(e^{\tilde{t}\hat{\omega}}-e^{(\tilde{t}-s)\hat{\omega}}\right)\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^2}{2\hat{\omega}^2e^{2(\tilde{t}-s)\hat{\omega}}\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^2}\right.\\
&\left[(\hat{\omega}-k)^2-(k+\hat{\omega})^2e^{\tilde{t}\hat{\omega}}e^{(\tilde{t}-s)\hat{\omega}}\right]+\frac{4k^2\theta^2}{\hat{\omega}^2}\frac{\left(e^{s\hat{\omega}}-1\right)^2\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^2}{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^2}\\
&+\frac{4k\theta e^{s\hat{\omega}}r_0}{\hat{\omega}}\frac{\left(e^{s\hat{\omega}}-1\right)\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^3}{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^3}\\
&\left.-2r_0\frac{\left[(\hat{\omega}-k)-(k+\hat{\omega})e^{\tilde{t}\hat{\omega}}\right]\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^4}{4\hat{\omega}e^{\tilde{t}\hat{\omega}}e^{-2s\hat{\omega}}\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^3}\right]\\
&+\frac{1}{2}\tilde{G}_{\alpha}^{(1)}(0,s,\tilde{t})\left[\frac{2e^{s\hat{\omega}}r_0\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]^2}{\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{\tilde{t}\hat{\omega}}-1\right)\right]^2}\right.\\
&\left.+\frac{\left[(\hat{\omega}-k)-(k+\hat{\omega})e^{(\tilde{t}-s)\hat{\omega}}\right]\left[2\hat{\omega}+(\hat{k}+\hat{\omega})\left(e^{(\tilde{t}-s)\hat{\omega}}-1\right)\right]}{4\hat{\omega}e^{(\tilde{t}-s)\hat{\omega}}}\right].
\end{aligned}
$$

Figure 4.11 shows the value of $\mathrm{V}ar[M_s]$ based on CIR model. As displayed in Figure 4.11 and 4.12, the time where $\mathrm{V}ar[M_s]$ reaches the maximum occurs at the early stage of the contract life. The utility function, described in (3.7), is applied to CIR model, where the value of the utility function is presented in Figure 4.13 and 4.14. Vasicek model is consistent with the optimal refinancing time, obtained by the optimization of the utility function will decrease with the increase of k or σ^2. Table A.5 shows that the optimal time will decrease when ρ increases. Again, when $\rho = 1$, the optimal refinancing time will be shortest.

4.3.3 $T = \infty$

The value of $\mathrm{E}[M_s]$ can be evaluated as followings with $T = \infty$.

$$
\begin{aligned}
&\mathrm{E}[M_s] \\
=&P(0)\int_s^\infty \frac{-(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)+(\omega+k)^2e^{2t\omega-s\omega}\left(r_0-\frac{k\theta}{\omega}\right)-\frac{k\theta}{\omega}(\omega^2-k^2)\left(e^{(t-s)\omega}+e^{t\omega}\right)}{\left[2\omega+(k+\omega)\left(e^{t\omega}-1\right)\right]^2}B(0,t)dt \\
=&P(0)\int_s^\infty \frac{-(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)e^{-2t\omega}+(\omega+k)^2e^{-s\omega}\left(r_0-\frac{k\theta}{\omega}\right)-\frac{k\theta}{\omega}(\omega^2-k^2)\left(e^{-(t+s)\omega}+e^{-t\omega}\right)}{\left[2\omega e^{-t\omega}+(k+\omega)\left(1-e^{-t\omega}\right)\right]^2} \\
&B(0,t)dt,
\end{aligned}
$$

which is convergent due to the fact that $B(0,\infty)=0$.

The value of $\mathrm{V}ar[M_s]$ is

$$
\begin{aligned}
&\mathrm{V}ar[M_s] \\
=&2P(0)^2\int_s^\infty\int_s^t\Bigg\{r_0^2\tilde{G}(0,s,\tilde{t})-2r_0\tilde{G}_\alpha^{(1)}(0,s,\tilde{t})+\tilde{G}_\alpha^{(2)}(0,s,\tilde{t})dhdt \\
-&\frac{-(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)e^{-2t\omega}+(\omega+k)^2e^{-s\omega}\left(r_0-\frac{k\theta}{\omega}\right)-\frac{k\theta}{\omega}(\omega^2-k^2)\left(e^{-(t+s)\omega}+e^{-t\omega}\right)}{\left[2\omega e^{-t\omega}+(k+\omega)\left(1-e^{-t\omega}\right)\right]^2}B(0,t) \\
&\frac{-(\omega-k)^2\left(r_0+\frac{k\theta}{\omega}\right)e^{-2h\omega}+(\omega+k)^2e^{-s\omega}\left(r_0-\frac{k\theta}{\omega}\right)-\frac{k\theta}{\omega}(\omega^2-k^2)\left(e^{-(h+s)\omega}+e^{-h\omega}\right)}{\left[2\omega e^{-h\omega}+(k+\omega)\left(1-e^{-h\omega}\right)\right]^2}B(0,h)\Bigg\},
\end{aligned}
$$

where the values of $\tilde{G}(0,s,\tilde{t})$, $\tilde{G}_\alpha^{(1)}(0,s,\tilde{t})$ and $\tilde{G}_\alpha^{(2)}(0,s,\tilde{t})$ are given in Section 4.3.2.

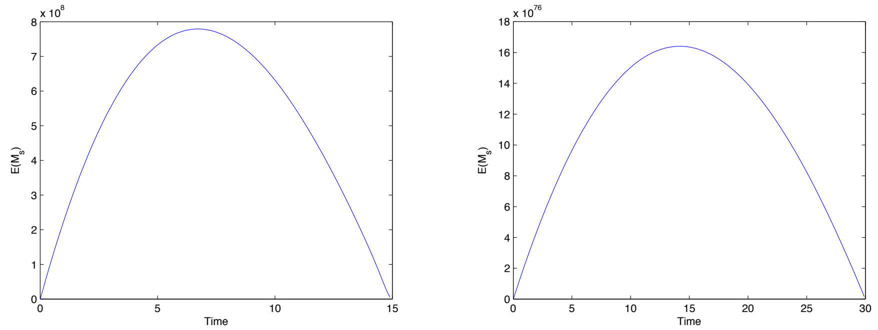

Figure 4.1: The numerical value of $\mathrm{E}[M_s]$ under Merton model when $T = 15$ and $T = 30$, where the value of the parameters are $u = -0.001$ and $\sigma^2 = 0.04$.

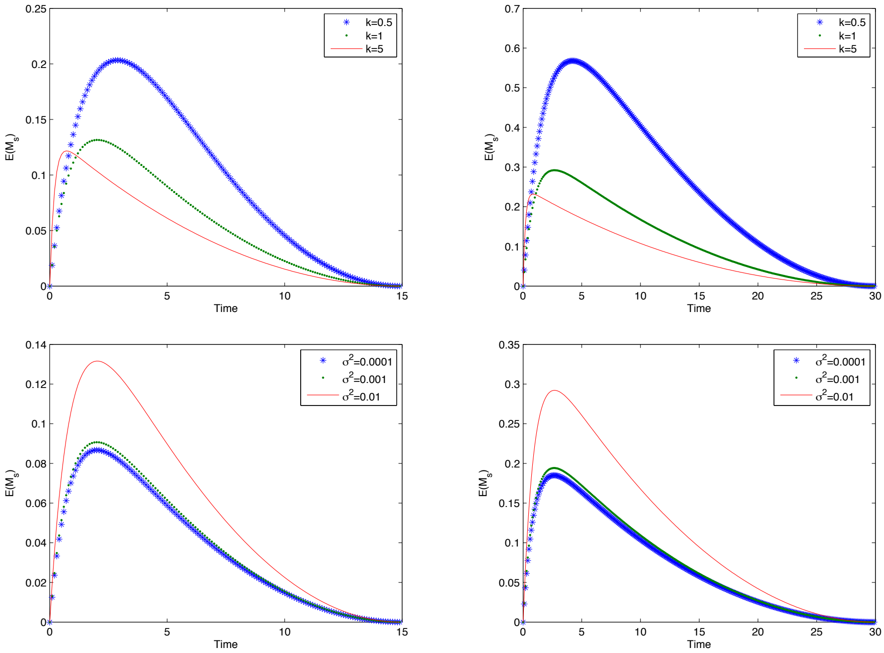

Figure 4.2: The numerical value of $\mathrm{E}[M_s]$ with different parameters under Vasicek model.

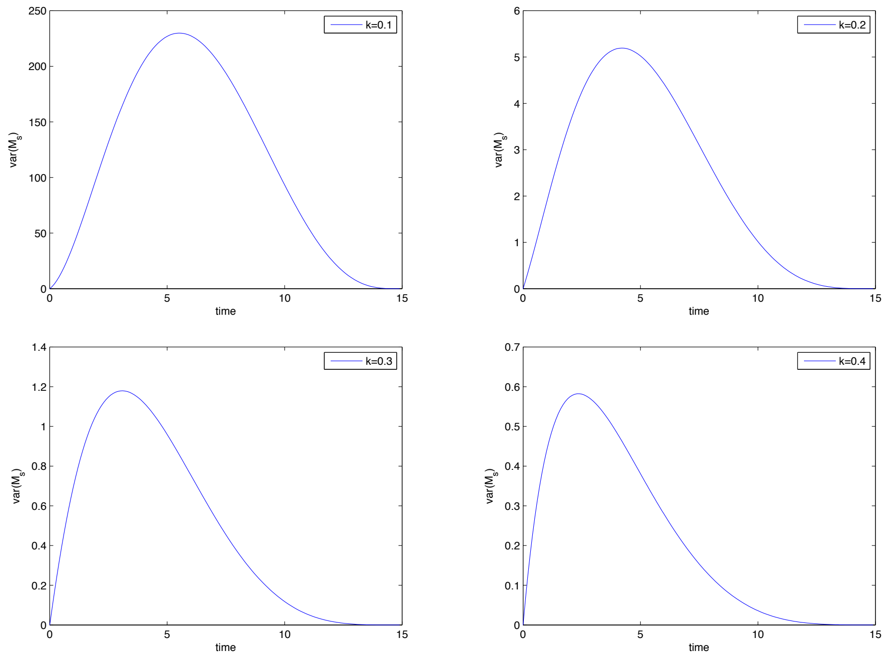

Figure 4.3: The numerical value of $\mathrm{V}ar[M_s]$ with different k under Vasicek model.

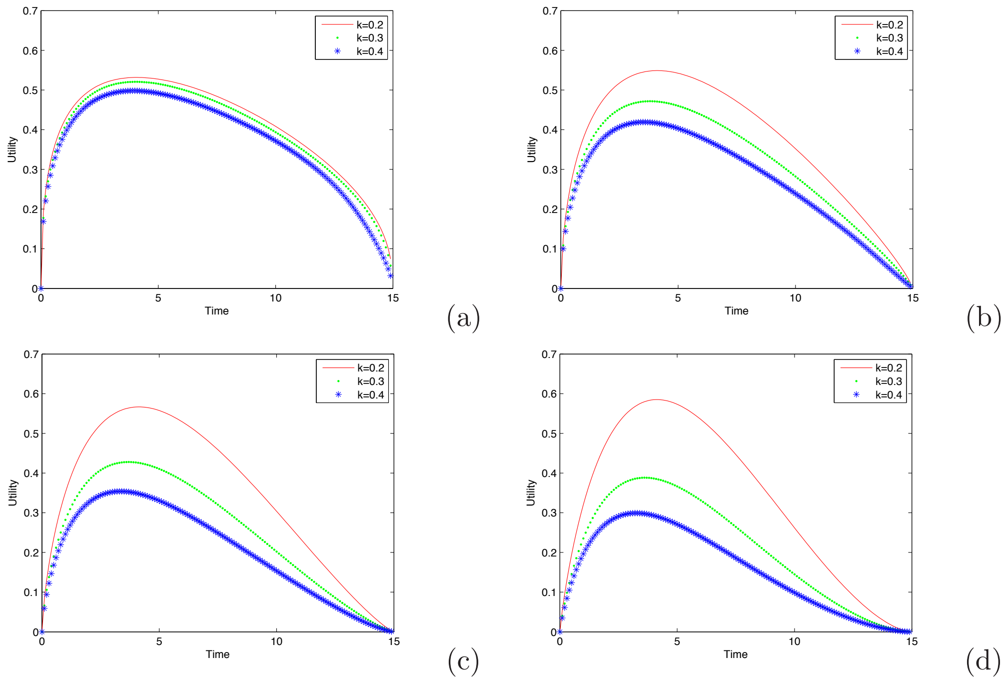

Figure 4.4: The numerical value of $U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{Var[M_s]}}\right)$ under Vasicek model with (a)$\rho = 0.6$, (b)$\rho = 0.7$, (c)$\rho = 0.8$, (d)$\rho = 0.9$.

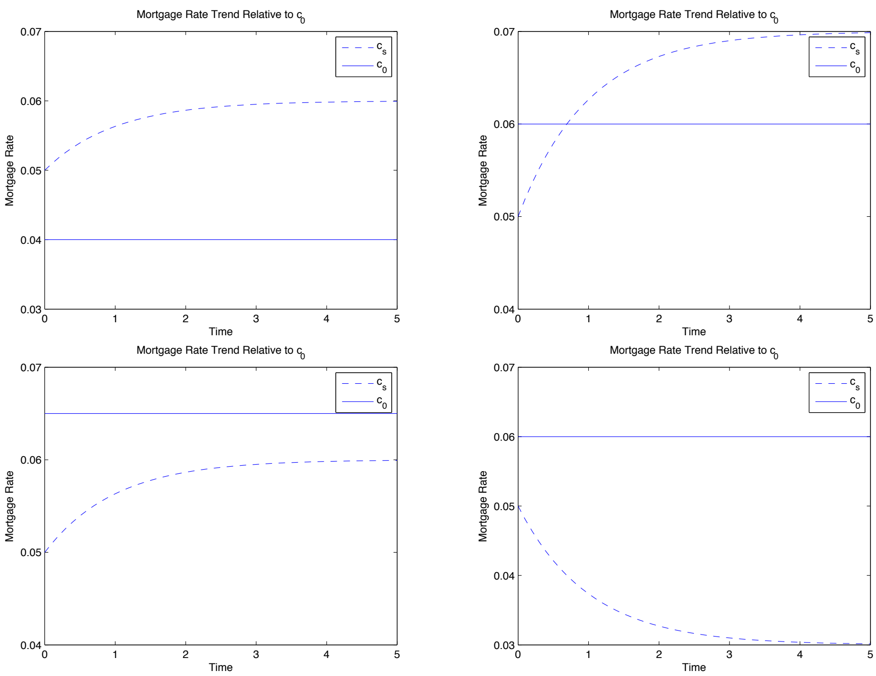

Figure 4.5: Comparison Between c_0 and Mortgage Rate Process for Small Volatility.

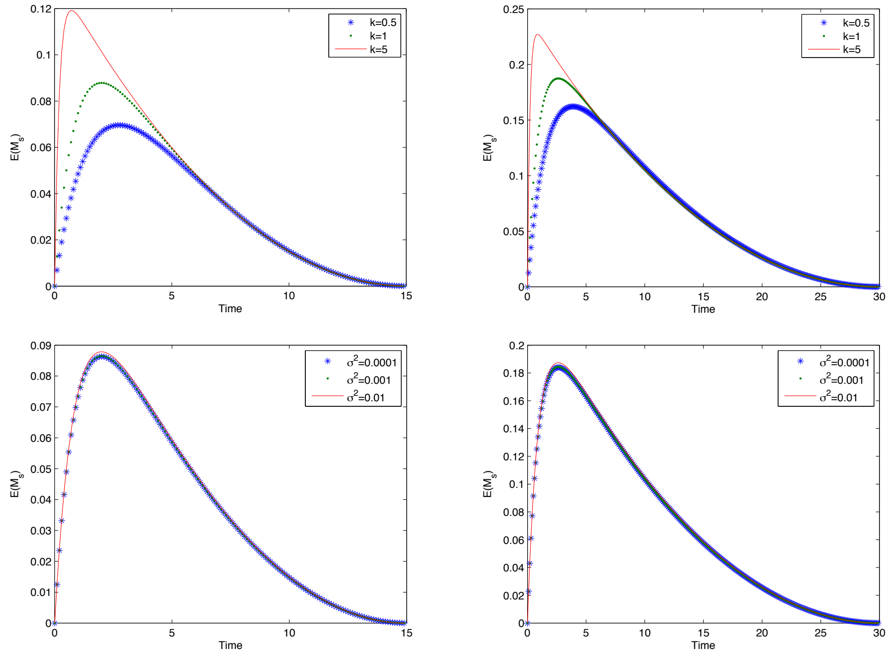

Figure 4.6: The numerical value of E$[M_s]$ with different parameters under CIR model.

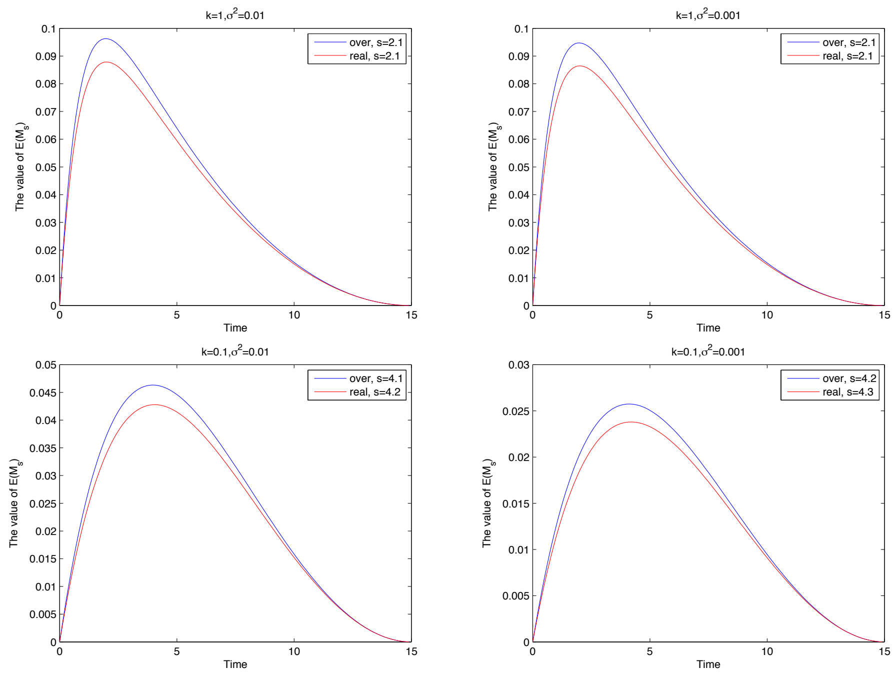

Figure 4.7: The numerical value of E$[M_s]$ through different approximating methods under CIR model.

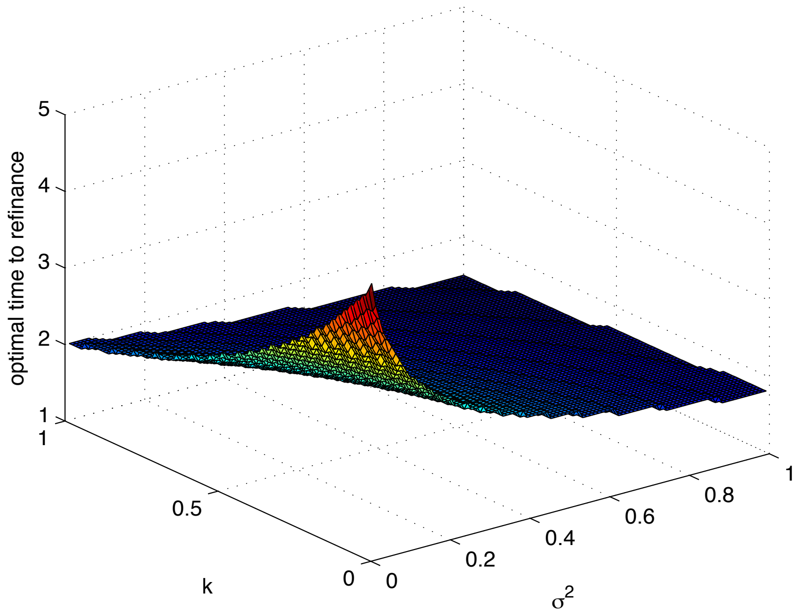

Figure 4.8: The 3D Graph of Optimal Refinancing Time ($\rho = 1$) with the Change of k and σ^2 under CIR model.

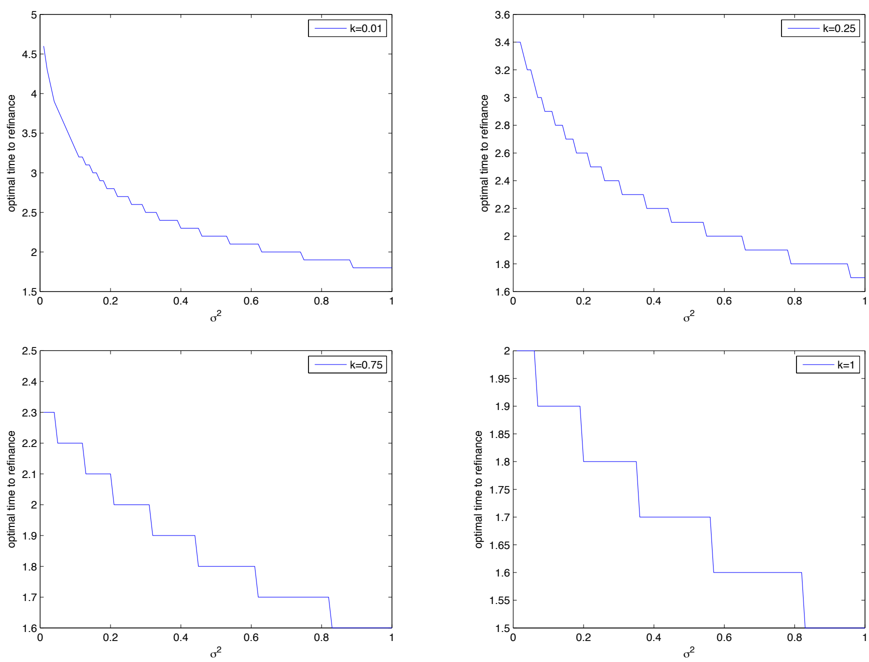

Figure 4.9: The Optimal Refinancing Time ($\rho = 1$) with the Change of k under CIR model.

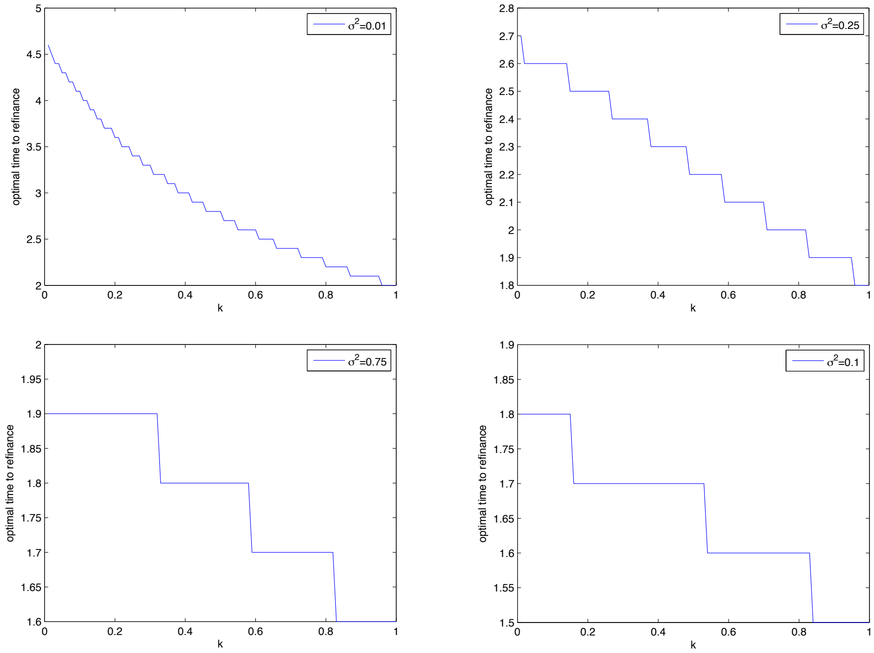

Figure 4.10: The Optimal Refinancing Time ($\rho = 1$) with the Change of σ^2 under CIR model.

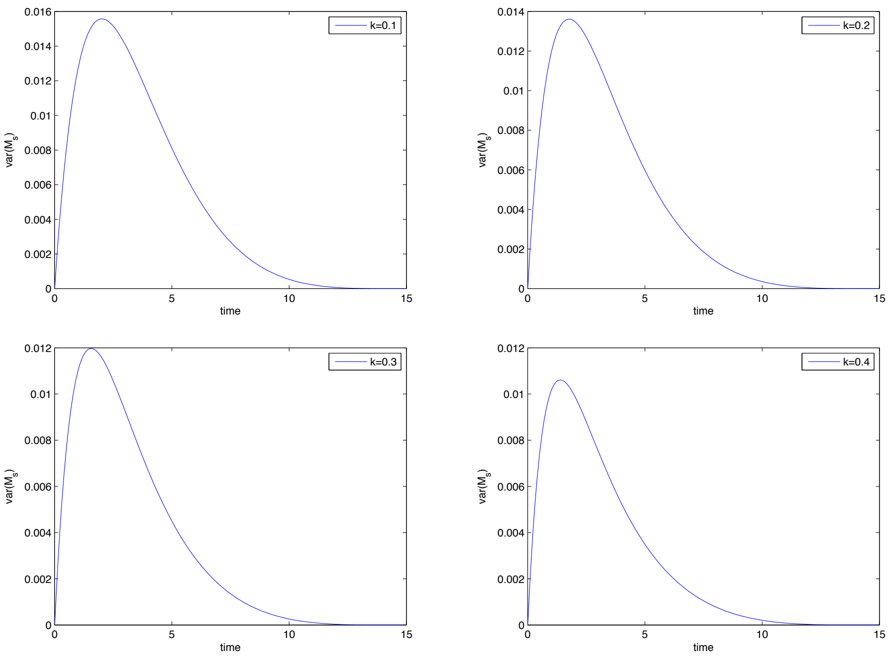

Figure 4.11: The numerical value of $\mathrm{V}ar[M_s]$ with different k under CIR model.

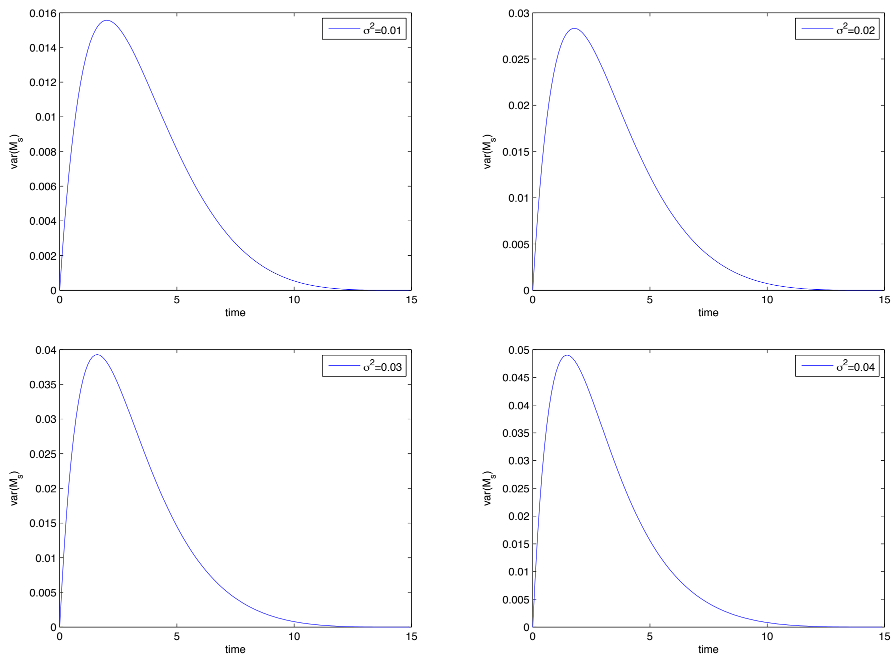

Figure 4.12: The numerical value of $\mathrm{V}ar[M_s]$ with different σ^2 under CIR model.

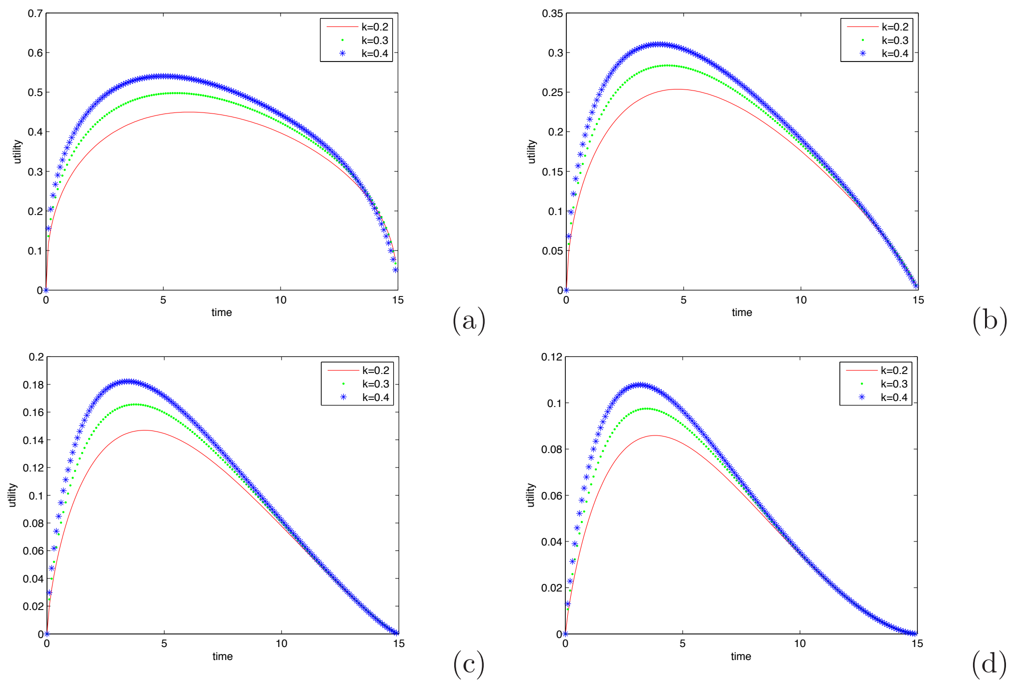

Figure 4.13: The numerical value of $U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{\mathrm{V}ar[M_s]}}\right)$ under CIR model with (a)$\rho = 0.6$, (b)$\rho = 0.7$, (c)$\rho = 0.8$, (d)$\rho = 0.9$ by the variation of k.

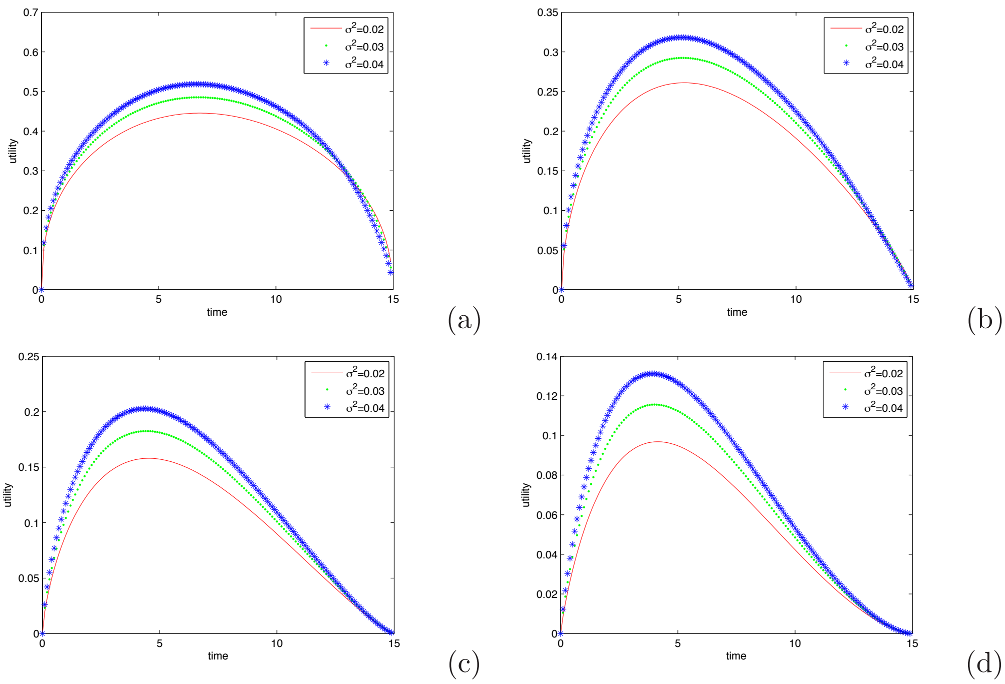

Figure 4.14: The numerical value of $U\left(\mathrm{E}[M_s], \frac{1}{\sqrt{\mathrm{V}ar[M_s]}}\right)$ under CIR model with (a)$\rho = 0.6$, (b)$\rho = 0.7$, (c)$\rho = 0.8$, (d)$\rho = 0.9$ by the variation of σ^2.

Chapter 5

Special Case: $\sigma = 0$

The special cases in this section give an intuitional view of our problem. We assume r_t is a deterministic decreasing function of t, and recall that

$$\begin{aligned} M_s =& P(0)\left[\frac{c_0}{1-e^{-c_0 T}} - \frac{c_s\left[1-e^{-c_0(T-s)}\right]}{\left[1-e^{-r_0 T}\right]\left[1-e^{-r_s(T-s)}\right]}\right]\int_s^T e^{-R(t)}dt \\ =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0 T}\right]}(r_0-r_s)\int_s^T e^{-R(t)}dt, \end{aligned} \tag{5.1}$$

where $R(t) = \int_0^t r_v dv$. The optimal time will be obtained by $\frac{dM_s}{ds} = 0$.

5.1 r_t is a decreasing linear function

5.1.1 $T < \infty$

When $\sigma = 0$ in (4.1), we obtain a linear function of r_t, where

$$r_t = r_0 + ut. \tag{5.2}$$

Note that in this case, the parameter u should be negative. Otherwise, r_t will be an increasing function and there is no point of refinancing. Thus, we may let $u_1 = -u > 0$, and thus $r_t = r_0 - u_1 t$. Then we rewrite (5.1) as

$$\begin{aligned} M_s =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0 T}\right]}u_1 s\int_s^T e^{-\int_0^t r_0 - u_1 v dv}dt \\ =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0 T}\right]}u_1 s\int_s^T e^{-r_0 t+\frac{1}{2}u_1 t^2}dt \\ =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0 T}\right]}u_1 s\sqrt{\frac{\pi}{2u_1}}e^{-\frac{r_0^2}{2u_1}}\left[\text{erfi}\left(\frac{u_1 T - r_0}{\sqrt{2u_1}}\right) - \text{erfi}\left(\frac{u_1 s - r_0}{\sqrt{2u_1}}\right)\right], \end{aligned} \tag{5.3}$$

where $erfi(z)$ gives the imaginary error function $\frac{erf(iz)}{i}$. The optimal time will given by s satisfying

$$\begin{aligned} &\left[\frac{(r_0(T-s)+1)\,e^{-r_0(T-s)}-1}{(T-s)\left[e^{-r_0(T-s)}+r_0(T-s)-1\right]} + \frac{1}{s}\right]\sqrt{\frac{\pi}{2u_1}}e^{-\frac{r_0^2}{2u_1}}\left[\text{erfi}\left(\frac{u_1 T - r_0}{\sqrt{2u_1}}\right) - \text{erfi}\left(\frac{u_1 s - r_0}{\sqrt{2u_1}}\right)\right] \\ =& e^{-r_0 s+\frac{1}{2}u_1 s^2}. \end{aligned}$$

From Table A.6, we can see the nonconvergent result of s when T increases. Moreover, the value of M_s will be infinite when $T = \infty$.

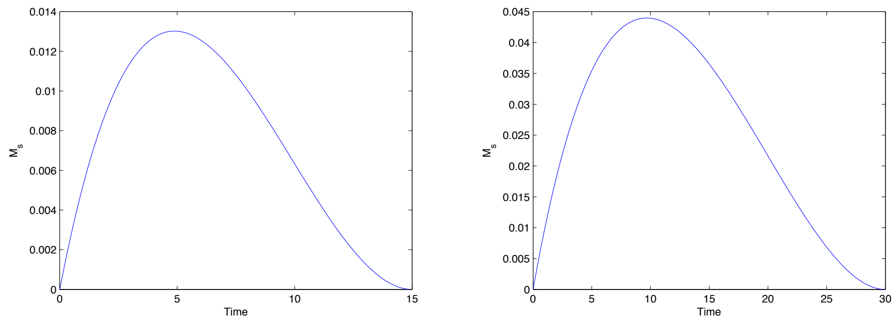

Figure 5.1: The numerical value of M_s under Merton model with $\sigma = 0$ when $T = 15$ and $T = 30$, with $u_1 = 0.001$.

5.1.2 $T = \infty$

We have the following formula for M_s when $T = \infty$, where

$$M_s = P(0) u_1 s \int_s^{\infty} e^{-r_0 t + \frac{1}{2} u_1 t^2} dt. \tag{5.4}$$

Adopting $\frac{dM_s}{ds} = 0$ gives

$$\sqrt{\frac{\pi}{2u_1}} e^{-\frac{r_0^2}{2u_1}} \left[\operatorname{erfi}\left(\frac{u_1 T - r_0}{\sqrt{2u_1}} \right) - \operatorname{erfi}\left(\frac{u_1 s - r_0}{\sqrt{2u_1}} \right) \right] = s e^{-r_0 s + \frac{1}{2} u_1 s^2}.$$

However, the value of M_s will be infinite as $T = \infty$, leading to an nonconvergent result of the optimal time.

5.2 r_t is a piecewise function

5.2.1 $T < \infty$

We assume r_t is a piecewise function with

$$r_t = \begin{cases} r_0 & \text{for} \quad s < s^\star \\ r_1 & \text{for} \quad s \geqslant s^\star, \end{cases} \tag{5.5}$$

where $r_1 < r_0$.

Thus, M_s can be described as

$$\begin{aligned} M_s =& P(0) \frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1 - e^{-r_0 T}\right]} (r_0 - r_1) \int_s^T e^{-r_1 t - (r_0 - r_1)s^\star} dt \\ =& P(0) \frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{r_0(T-s)\left[1 - e^{-r_0 T}\right]} \frac{r_0 - r_1}{r_1} e^{-(r_0 - r_1)s^\star} \left(e^{-r_1 s} - e^{-r_1 T} \right), \end{aligned} \tag{5.6}$$

with $s \in [s^\star, T)$.

We let $f(s) = \frac{e^{-r_0(T-s)}+r_0(T-s)-1}{T-s}\left(e^{-r_1 s} - e^{-r_1 T}\right)$, and we have

$$
\begin{aligned}
f'(s) =& \frac{\left(r_0 e^{-r_0(T-s)} - r_0\right)(T-s) + e^{-r_0(T-s)} + r_0(T-s) - 1}{(T-s)^2}\left(e^{-r_1 s} - e^{-r_1 T}\right) \\
& - r_1 e^{-r_1 s}\frac{e^{-r_0(T-s)} + r_0(T-s) - 1}{T-s} \\
=& \frac{r_0\left(e^{-r_0(T-s)} - 1\right)}{T-s} + \frac{\left[e^{-r_1 s} - e^{-r_1 T} - r_1 e^{-r_1 s}(T-s)\right]\left[e^{-r_0(T-s)} + r_0(T-s) - 1\right]}{(T-s)^2}.
\end{aligned}
$$

We let $f_1(s) = e^{-r_1 s} - e^{-r_1 T} - r_1 e^{-r_1 s}(T-s)$ and $f_2(s) = e^{-r_0(T-s)} + r_0(T-s) - 1$, as

$$
\begin{aligned}
f_1'(s) =& -r_1 e^{-r_1 s} + r_1^2 e^{-r_1 s}(T-s) + r_1 e^{-r_1 s} = r_1^2 e^{-r_1 s}(T-s) > 0 \\
f_2'(s) =& r_0 e^{-r_0(T-s)} - r_0 = r_0\left(e^{-r_0(T-s)} - 1\right) < 0,
\end{aligned}
$$

we have $f_1(s) < f_1(T) = 0$ and $f_2(s) > f_2(T) = 0$. Thus, we have $f'(s) < 0$.

We can see that M_s is a decreasing function with $s \in [s^\star, T)$, thus, the maximum value will be occurred at $s = s^\star$, with

$$
M_{s^\star} = P(0)\frac{e^{-r_0(T-s^\star)} + r_0(T-s^\star) - 1}{r_0(T-s^\star)\left[1 - e^{-r_0 T}\right]}\frac{r_0 - r_1}{r_1}e^{-(r_0-r_1)s^\star}\left(e^{-r_1 s^\star} - e^{-r_1 T}\right).
$$

Figure 5.2 displays the value of M_s with the parameters $P(0) = 1$, $r_0 = 0.05$, $r_1 = 0.03$, $T = 15$ and $s^\star = 3.5$. As the interest rate will be constant after a sudden change at $s^\star$, the debtor will lose benefit if he or she does not refinance at time $t = s^\star$, due to the fact that the debtor will pay more interests in the future.

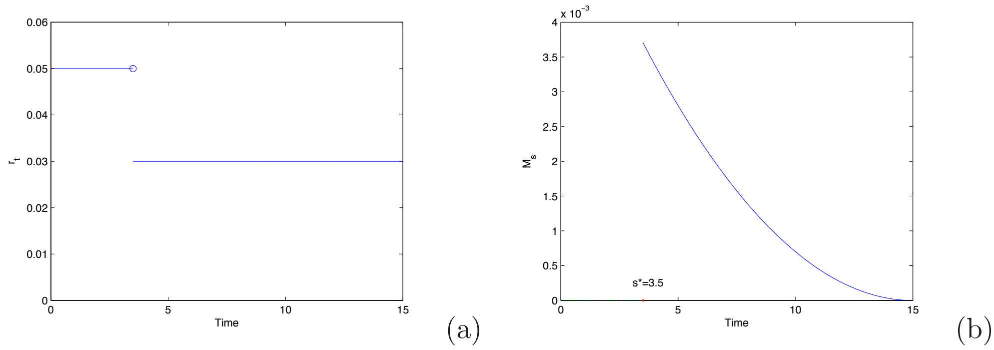

Figure 5.2: (a)The function of r_t. (b)The value of M_s when r_t is a piecewise function.

If we assume

$$
r_t = \begin{cases} r_0 & \text{for} \quad s < s^\star \\ r_s & \text{for} \quad s \geqslant s^\star \end{cases} \tag{5.7}
$$

where r_s is a decreasing function of s, with the initial value of r_1. We are interested in the value of r_s, where the profit by refinancing at time s equals a value C. As we have

$$\begin{aligned} M_s =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}(r_0-r_s)\int_s^T e^{-\int_0^{s^\star} r_0 dv}e^{-\int_{s^\star}^t r_v dv}dt \\ =& P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}(r_0-r_s)e^{-r_0 s^\star}\int_s^T e^{-\int_{s^\star}^t r_v dv}dt. \end{aligned} \tag{5.8}$$

Thus, r_s can be solved by

$$\int_s^T e^{-\int_{s^\star}^t r_v dv}dt = \frac{Cr_0(T-s)e^{r_0 s^\star}\left[1-e^{-r_0T}\right]}{P(0)\left[e^{-r_0(T-s)}+r_0(T-s)-1\right](r_0-r_s)}$$

We let $f(s,r_s) = \frac{Cr_0(T-s)e^{r_0 s^\star}\left[1-e^{-r_0T}\right]}{P(0)\left[e^{-r_0(T-s)}+r_0(T-s)-1\right](r_0-r_s)}$, such that

$$f(s,r_s) = \int_s^T e^{-\int_{s^\star}^t r_v dv}dt, \tag{5.9}$$

Taking derivative of (5.9) with respect of s for both sides gives

$$-\frac{d}{ds}f(s,r_s) = e^{-\int_{s^\star}^s r_v dv},$$

or equivalently,

$$\ln\left[-\frac{d}{ds}f(s,r_s)\right] = -\int_{s^\star}^s r_v dv,$$

We can obtain the formula for r_s by taking the derivative of the above equation with respect to s, which implies

$$r_s = -\frac{d}{ds}\ln\left[-\frac{d}{ds}f(s,r_s)\right] = -\frac{\frac{d^2}{ds^2}f(s,r_s)}{\frac{d}{ds}f(s,r_s)}.$$

Hence, r_s can be solved by the following ODE

$$\begin{aligned} r_s =& \frac{(T-s)\left[-r_0^2 e^{-r_0(T-s)}+2r_0\left(e^{-r_0(T-s)}-1\right)\frac{dr_s}{ds}+\left(e^{-r_0(T-s)}+r_0(T-s)-1\right)\frac{d^2r_s}{ds^2}\right]}{\left(e^{-r_0(T-s)}+r_0(T-s)-1\right)\left(r_0-r_s-(T-s)\frac{dr_s}{ds}\right)+(T-s)r_0\left(e^{-r_0(T-s)}-1\right)(r_0-r_s)} \\ &+2\frac{r_0\left(e^{-r_0(T-s)}-1\right)(r_0-r_s)-\left(e^{-r_0(T-s)}+r_0(T-s)-1\right)\frac{dr_s}{ds}}{\left(e^{-r_0(T-s)}+r_0(T-s)-1\right)(r_0-r_s)}, \end{aligned}$$

with the initial boundary of $r_{s^\star}=r_1$, $\frac{dr_s}{ds}|_{s^\star}=\gamma<0$.

Figure 5.3 displays the relation between the optimal refinancing rate and the refinancing time. The curve of r_s can be defined as the optimal boundary to refinance. The debtor will refinance when the market interest rate hits the curve. If the market interest rate is on the boundary, the debtor is indifferent to the refinancing time as the profit will stay at C.

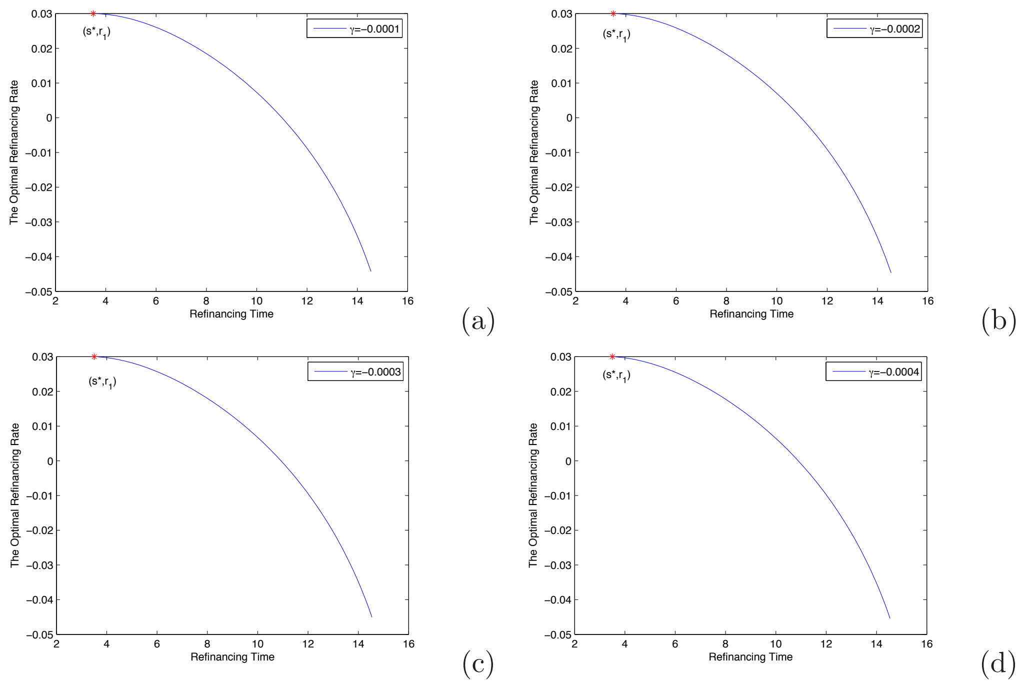

Figure 5.3: The relation between the optimal refinancing rate and the refinancing time.

5.2.2 $T = \infty$

As we have

$$
\begin{aligned}
M_s =& P(0)(r_0 - r_1)\int_s^{\infty} e^{-r_1 t - (r_0 - r_1)s^\star} dt \\
=& P(0)\frac{r_0 - r_1}{r_1} e^{-(r_0 - r_1)s^\star} e^{-r_1 s},
\end{aligned}
\tag{5.10}
$$

with $s \in [s^\star, T)$. It is clearly that M_s will reach the maximum point when $s \in [s^\star, T)$. And the profit, M_s depends on the initial principle, the change of the interest rate and the discount rate.

5.3 r_t is a decreasing exponential function

When the volatility for the risk-free rate is negligible, the stochastic integral solution to the CIR or Vasicek stochastic differential equation can be approximated by the following deterministic functions. That is,

$$dr_t = k(\theta - r_t)dt,$$

or equivalently,

$$r_t = \theta + (r_0 - \theta)e^{-kt}. \tag{5.11}$$

5.3.1 $T < \infty$

$$\begin{aligned}M_s =&P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}(r_0-r_s)\int_s^T e^{-\frac{r_0-\theta}{k}\left(1-e^{-kt}\right)-\theta t}dt\\ =&P(0)\frac{e^{-r_0(T-s)}+r_0(T-s)-1}{r_0(T-s)\left[1-e^{-r_0T}\right]}(r_0-\theta)\left(1-e^{-ks}\right)\int_s^T e^{-\frac{r_0-\theta}{k}\left(1-e^{-kt}\right)-\theta t}dt.\end{aligned}$$

As $\frac{d}{ds}M_s = 0$, we have

$$\begin{aligned}&\left[\frac{\left(r_0(T-s)+1\right)e^{-r_0(T-s)}-1}{(T-s)\left[e^{-r_0(T-s)}+r_0(T-s)-1\right]}+\frac{ke^{-ks}}{1-e^{-ks}}\right]\int_s^T e^{-\frac{r_0-\theta}{k}\left(1-e^{-kt}\right)-\theta t}dt\\ =&e^{-\frac{r_0-\theta}{k}\left(1-e^{-ks}\right)-\theta s}. \end{aligned} \tag{5.12}$$

We use the numerical solution to obtain the optimal refinancing time based on (5.12). The result in Table A.7 shows the optimal time to refinance increases as the contractual time T increases. However, with large T, the optimal time to refinance stays at a stable point.

5.3.2 $T = \infty$

We have

$$M_s = P(0)(r_0-\theta)\left(1-e^{-ks}\right)\int_s^\infty e^{-\frac{r_0-\theta}{k}\left(1-e^{-kt}\right)-\theta t}dt.$$

We let $x = e^{-ks}$, $y = e^{-kt}$, and M_s can be continued as

$$\begin{aligned}M_s =&-P(0)\frac{r_0-\theta}{k}(1-x)\int_x^0 e^{-\frac{r_0-\theta}{k}(1-y)}y^{\frac{\theta}{k}-1}dy\\ =&P(0)\frac{r_0-\theta}{k}e^{-\frac{r_0-\theta}{k}}(1-x)\int_0^x e^{\frac{r_0-\theta}{k}y}y^{\frac{\theta}{k}-1}dy.\end{aligned}$$

To obtain the maximum/minimun point, we let

$$\frac{d}{dx}M_s = -P(0)\frac{r_0-\theta}{k}e^{-\frac{r_0-\theta}{k}}\int_0^x e^{\frac{r_0-\theta}{k}y}y^{\frac{\theta}{k}-1}dy + P(0)\frac{r_0-\theta}{k}e^{-\frac{r_0-\theta}{k}}(1-x)e^{\frac{r_0-\theta}{k}x}x^{\frac{\theta}{k}-1} = 0.$$

which gives

$$\int_0^x e^{\frac{r_0-\theta}{k}y}y^{\frac{\theta}{k}-1}dy = (1-x)e^{\frac{r_0-\theta}{k}x}x^{\frac{\theta}{k}-1}.$$

We let $y = xv$, the equation can be simplified as

$$\int_0^1 e^{\frac{r_0-\theta}{k}xv}x^{\frac{\theta}{k}}v^{\frac{\theta}{k}-1}dv = (1-x)e^{\frac{r_0-\theta}{k}x}x^{\frac{\theta}{k}-1}.$$

We always assume that $r_0 > \theta$, where the rational debtor will make profit by refinancing. To simplify the equation, we let $\alpha = \frac{r_0-\theta}{k} > 0$, then

$$x\int_0^1 e^{\alpha x(v-1)}v^{\frac{\theta}{k}-1}dv = 1-x.$$

We let $u = 1 - v$, then

$$\begin{aligned}
g(x) =& x \int_0^1 e^{\alpha x(v-1)} v^{\frac{\theta}{k}-1} dv \\
=& x \int_0^1 e^{\alpha x u} (1-u)^{\frac{\theta}{k}-1} du \\
=& x \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n x^n}{n!} \int_0^1 u^n (1-u)^{\frac{\theta}{k}-1} du \\
=& x\Gamma\left(\frac{\theta}{k}\right) \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n x^n}{\Gamma\left(n+1+\frac{\theta}{k}\right)} \\
=& 1 - x.
\end{aligned} \tag{5.13}$$

As $g(x)$ is an alternating series, we have

$$\begin{aligned}
x\Gamma\left(\frac{\theta}{k}\right) \sum_{n=0}^{2} \frac{(-1)^n \alpha^n x^n}{\Gamma\left(n+1+\frac{\theta}{k}\right)} >& 1 - x \\
x\Gamma\left(\frac{\theta}{k}\right) \sum_{n=0}^{3} \frac{(-1)^n \alpha^n x^n}{\Gamma\left(n+1+\frac{\theta}{k}\right)} <& 1 - x.
\end{aligned}$$

Then we can approximate $1 - x$ as

$$\frac{1}{2} x\Gamma\left(\frac{\theta}{k}\right) \sum_{n=0}^{2} \frac{(-1)^n \alpha^n x^n}{\Gamma\left(n+1+\frac{\theta}{k}\right)} + \frac{1}{2} x\Gamma\left(\frac{\theta}{k}\right) \sum_{n=0}^{3} \frac{(-1)^n \alpha^n x^n}{\Gamma\left(n+1+\frac{\theta}{k}\right)} = 1 - x,$$

which can be simplified as

$$\frac{1}{2} \frac{\alpha^3}{\frac{\theta}{k}\left(\frac{\theta}{k}+1\right)\left(\frac{\theta}{k}+2\right)\left(\frac{\theta}{k}+3\right)} x^4 - \frac{\alpha^2}{\frac{\theta}{k}\left(\frac{\theta}{k}+1\right)\left(\frac{\theta}{k}+2\right)} x^3 + \frac{\alpha}{\frac{\theta}{k}\left(\frac{\theta}{k}+1\right)} x^2 - \left[\frac{1}{\frac{\theta}{k}} + 1\right] x + 1 = 0.$$

We consider the special when $\frac{\theta}{k} = 1$, such that (5.13) can be rearranged as

$$x \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n x^n}{(n+1)!} = 1 - x,$$

which can be simplified as

$$e^{-\alpha x} - \alpha x + \alpha - 1 = 0,$$

with the solution

$$x = \frac{W(e^{1-\alpha}) + \alpha - 1}{\alpha}.$$

where $W(z)$ is the product log function.

Thus

$$s = -\frac{\ln\left(W(e^{1-\alpha}) + \alpha - 1\right) - \ln(\alpha)}{k}.$$

Chapter 6

Remarks and Future Work

This work examines the debtor's optimal refinancing strategy under restriction that only one refinancing opportunity is allowed across the duration of a mortgage loan. The numerical solution suggests the optimal refinancing time is more likely to appear at the early stage of the contract. The values of the utility as a function of time are generated, and the properties of which are analyzed and interpreted with real financial implications.

The current paper overcomes several weaknesses and implicit premises requiring both theoretical fortification and numerical enhancement in a recently developed seminal work on the optimal refinancing strategy (see [48]). The current work provide a complete and rigorous optimization formulation to the concerned problem.

Adopting the optimization of the utility function approach developed in this thesis, the analytic formulae are presented for the affine interest models. The obvious future work would be to present a general solution applicable to other interest rate models. As closed form solutions are not available for the multiple refinancing problems, a vital area for future research would be to introduce a model for the multiple refinancing problem. In addition, the validation of the theory on real data will be tested in the future.

Appendix A

Appendix of Tables

Table A.1: The Relative Error of the approximation in Lemma 3.2.3, with $y = W(-ae^{-a})+a$, where $W(z)$ is the product log function and $a = \frac{x_0}{1-e^{-x_0}} + \frac{1-e^{-x_0}-x_0}{(1-e^{-x_0})x_0}(x_0 - x)$ with $x_0 = 1.5$.

x	y	$y-x$	$\frac{y-x}{x}$
0.1	0.1216	0.0216	0.2160
0.2	0.2387	0.0387	0.1935
0.3	0.3518	0.0518	0.1727
0.4	0.4611	0.0611	0.1528
0.5	0.5672	0.0672	0.1344
0.6	0.6703	0.0703	0.1172
0.7	0.7707	0.0707	0.1010
0.8	0.8687	0.0687	0.0859
0.9	0.9643	0.0643	0.0714
1.0	1.0579	0.0579	0.0579
1.1	1.1496	0.0496	0.0451
1.2	1.2396	0.0396	0.0330
1.3	1.3279	0.0279	0.0215
1.4	1.4146	0.0146	0.0104
1.5	1.5	0	0

Table A.2: The Optimal time to Refinance (with $\rho = 1$) under Vasicek model. The value of parameters are $k = 1$, $\theta = 0.03$ and $\sigma^2 = 0.01$.

T	Optimal Time	T	Optimal Time
5	1.2	55	3.3
10	1.8	60	3.4
15	2.1	65	3.4
20	2.4	70	3.5
25	2.6	75	3.5
30	2.8	80	3.5
35	2.9	85	3.6
40	3	90	3.6
45	3.1	95	3.6
50	3.2	100	3.6

Table A.3: The Optimal time to Refinance under Vasicek model based on the utility function. The value of parameters are $k = 1$, $\theta = 0.03$, $\sigma^2 = 0.01$ and $T = 15$.

ρ	Optimal Time	ρ	Optimal Time
0.55	3.4	0.8	2.3
0.6	2.9	0.85	2.3
0.65	2.7	0.9	2.2
0.7	2.5	0.95	2.2
0.75	2.4	1	2.1

Table A.4: The Optimal time to Refinance (with $\rho = 1$) under CIR model. The value of parameters are $k = 1$, $\theta = 0.03$ and $\sigma^2 = 0.01$.

T	Optimal Time	T	Optimal Time
5	1.2	55	3.2
10	1.7	60	3.3
15	2.1	65	3.3
20	2.4	70	3.3
25	2.6	75	3.4
30	2.7	80	3.4
35	2.9	85	3.4
40	3	90	3.4
45	3.1	95	3.5
50	3.1	100	3.5

Table A.5: The Optimal time to Refinance under CIR model based on the utility function. The value of parameters are $k = 1$, $\theta = 0.03$, $\sigma^2 = 0.01$ and $T = 15$.

ρ	Optimal Time	ρ	Optimal Time
0.55	3.9	0.8	2.4
0.6	3.3	0.85	2.3
0.65	2.9	0.9	2.2
0.7	2.7	0.95	2.2
0.75	2.5	1	2.1

Table A.6: The Optimal time to Refinance under Merton model, with $\sigma = 0$.

T	Optimal Time	T	Optimal Time
5	1.7	55	18.6
10	3.3	60	20.7
15	4.9	65	23.1
20	6.9	70	25.7
25	8.1	75	28.6
30	9.7	80	31.9
35	11.0	85	35.6
40	13.0	90	39.6
45	14.8	95	43.9
50	16.6	100	48.5

Table A.7: The Optimal time to Refinance under CIR or Vasicek Model when $\sigma = 0$.

T	Optimal Time	T	Optimal Time
5	1.1	55	3.1
10	1.6	60	2.2
15	2	65	3.2
20	2.2	70	3.3
25	2.4	75	3.3
30	2.6	80	3.3
35	2.7	85	3.4
40	2.8	90	3.4
45	2.9	95	3.4
50	3	100	3.4

Appendix B

Appendix: a new method to compute $B(s,t)$ under Vasicek model

In the Vasicek model, as we have

$$\int_s^t r_u du = \frac{\sigma(W_t - W_s) + k\theta(t-s) - r_t + r_s}{k},$$

thus, the bond price is

$$B(s,t) = \mathrm{E}\Big[e^{-\int_s^t r_u du}|_{r_s}\Big] = \mathrm{E}\Big[e^{-\frac{\sigma(W_t-W_s)+k\theta(t-s)-r_t+r_s}{k}}|_{r_s}\Big],$$

where r_s is the short rate. As

$$r_t = \theta + (r_s - \theta)e^{-k(t-s)} + \sigma e^{-kt}\int_s^t e^{ku}dW_u,$$

the calculation of bond price could be continued as

$$\begin{aligned}
=&\mathrm{E}\Bigg[e^{-\frac{\sigma(W_t-W_s)-k\theta(t-s)-\theta-(r_s-\theta)e^{-k(t-s)}-\sigma e^{-kt}\int_s^t e^{ku}dW_u+r_s}{k}}|_{r_s}\Bigg]\\
=&e^{-\frac{(r_s-\theta)\left(1-e^{-k(t-s)}\right)+k\theta(t-s)}{k}}\mathrm{E}\Bigg[e^{-\frac{\sigma(W_t-W_s)-\sigma e^{-kt}\int_s^t e^{ku}dW_u}{k}}|_{r_s}\Bigg]\\
=&e^{-\frac{(r_s-\theta)\left(1-e^{-k(t-s)}\right)+k\theta(t-s)}{k}}\mathrm{E}\Bigg[e^{-\frac{\sigma\int_s^t\left[1-e^{-k(t-u)}\right]dW_u}{k}}\Bigg],
\end{aligned}$$

and we let $f(u) = -\frac{\sigma}{k}\left(1-e^{-k(t-u)}\right)$, the above equation is

$$= e^{-\frac{(r_s-\theta)\left(1-e^{-k(t-s)}\right)+k\theta(t-s)}{k}}\mathrm{E}\Big[e^{\int_s^t f(u)dW_u}\Big]. \tag{B.1}$$

We suppose $q = \int_0^t f(u)dW_u$. It is well known that q follows normal distribution. Clearly, we have

$$\mathrm{E}[q] = 0,$$

and by adopting the Ito's isometry, we obtain

$$\begin{aligned}
\mathrm{Var}[q] =& \mathrm{E}[q^2] - [\mathrm{E}[q]]^2 \\
=& \mathrm{E}[q^2] \\
=& \mathrm{E}\left[\left(\int_s^t f(u)dW_u\right)^2\right] \\
=& \int_s^t [f(u)]^2\, dv \\
=& \frac{\sigma^2}{k^2}\left(t - s + 2\frac{e^{-k(t-s)}}{k} - \frac{e^{-2k(t-s)}}{2k} - \frac{3}{2k}\right),
\end{aligned}$$

which implies $q \sim N(0, \frac{\sigma^2}{k^2}\left(t - s + 2\frac{e^{-k(t-s)}}{k} - \frac{e^{-2k(t-s)}}{2k} - \frac{3}{2k}\right))$. Therefore, we have

$$\mathrm{E}\left[e^{\int_s^t f(u)dW_u}\right] = e^{\frac{\sigma^2}{2k^2}\left(t-s+2\frac{e^{-k(t-s)}}{k} - \frac{e^{-2k(t-s)}}{2k} - \frac{3}{2k}\right)}.$$

Thus, we continue calculating (B.1) as

$$\begin{aligned}
=& e^{-\frac{(r_s-\theta)\left(1-e^{-k(t-s)}\right)+k\theta(t-s)}{k}} e^{\frac{\sigma^2}{2k^2}\left(t-s+2\frac{e^{-k(t-s)}}{k} - \frac{e^{-2k(t-s)}}{2k} - \frac{3}{2k}\right)} \\
=& e^{-\frac{1-e^{-k(t-s)}}{k}r_s + \frac{1-e^{-k(t-s)}}{k}\theta - \left(\theta - \frac{\sigma^2}{2k^2}\right)(t-s) + \frac{\sigma^2}{2k^2}\left[-\frac{1-e^{-k(t-s)}}{k} + \frac{e^{-k(t-s)}}{k} - \frac{e^{-2k(t-s)}}{2k} - \frac{1}{2k}\right]} \\
=& e^{-\frac{1-e^{-k(t-s)}}{k}r_s - \left(\theta - \frac{\sigma^2}{2k^2}\right)(t-s) + \left(\theta - \frac{\sigma^2}{2k^2}\right)\frac{1-e^{-k(t-s)}}{k} + \frac{\sigma^2}{2k^2}\left[\frac{e^{-k(t-s)}}{k} - \frac{e^{-2k(t-s)}}{2k} - \frac{1}{2k}\right]} \\
=& e^{-\frac{1-e^{-k(t-s)}}{k}r_s - \left(\theta - \frac{\sigma^2}{2k^2}\right)(t-s) + \left(\theta - \frac{\sigma^2}{2k^2}\right)\frac{1-e^{-k(t-s)}}{k} - \frac{\sigma^2}{4k}\left(\frac{1-e^{-k(t-s)}}{k}\right)^2} \\
=& e^{\left(\theta - \frac{\sigma^2}{2k^2}\right)\left[\frac{1-e^{-k(t-s)}}{k} - (t-s)\right] - \frac{\sigma^2}{4k}\left(\frac{1-e^{-k(t-s)}}{k}\right)^2 - \frac{1-e^{-k(t-s)}}{k}r_s}.
\end{aligned}$$

If we let

$$\begin{aligned}
A_1(s,t) =& e^{\left(\theta - \frac{\sigma^2}{2k^2}\right)[A_2(s,t) - (t-s)] - \frac{\sigma^2}{4k}A_2^2(s,t)}. \\
A_2(s,t) =& \frac{1 - e^{-k(t-s)}}{k},
\end{aligned}$$

the bond price is

$$B(s,t) = \mathrm{E}\left[e^{-\int_s^t r_u du}\right] = A_1(s,t)e^{-A_2(s,t)r_2}.$$

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